Algebraic Properties and Uniform Computational Content

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Varieties of Algorithmic Information
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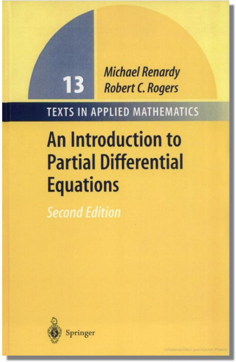
Outline

Mathematical Problems

Phenomenology of Mathematical Implication



Lemma 8.36. *The open mapping theorem, the bounded inverse theorem, and the closed graph theorem are equivalent.*

Phenomenology of Mathematical Implication

Proof. **Open mapping theorem $\Rightarrow$ bounded inverse theorem.** It is immediately clear from the hypotheses of the bounded inverse theorem that a linear inverse operator A^{-1} with domain $\mathcal{R}(A)$ exists. The nontrivial assertion is that A^{-1} is bounded. However, this follows from the open mapping theorem, the equivalence of boundedness and continuity for linear operators, and the topological version of the definition of continuity: that an operator T is continuous if and only if the inverse image of open sets in $\mathcal{R}(T)$ is open in $\mathcal{D}(T)$. (The inverse image of an open set in $\mathcal{R}(A^{-1})$ $(= X = \mathcal{D}(A))$ under A^{-1} is the same as the image of the set under A.)

Bounded inverse theorem $\Rightarrow$ closed graph theorem. We first observe that the product space $X \times Y$ is a Banach space with norm

$$\|(x, y)\| = \|x\| + \|y\|. \tag{8.31}$$

Our hypothesis is that $\Gamma(A)$ is a closed subspace in $X \times Y$ and $\mathcal{D}(A)$ is a closed subspace in X. Thus, $\Gamma(A)$ and $\mathcal{D}(A)$ are Banach spaces. We now define a projection map

$$P : \Gamma(A) \to \mathcal{D}(A) \tag{8.32}$$

by

$$P(x, Ax) := x. \tag{8.33}$$

Note that P is linear and bijective. If fact, its inverse

$$P^{-1} : \mathcal{D}(A) \to \Gamma(A) \tag{8.34}$$

is defined by

$$P^{-1}x := (x, Ax). \tag{8.35}$$

The mapping P is also bounded since

$$\|P(x, Ax)\| = \|x\| \leq \|x\| + \|Ax\| = \|(x, Ax)\|. \tag{8.36}$$

Thus, by the bounded inverse theorem (8.34) there is a constant C such that

$$\|(x, Ax)\| = \|P^{-1}x\| \leq C\|x\|. \tag{8.37}$$

But this implies A is bounded since

$$\|Ax\| \leq \|(x, Ax)\| \leq C\|x\| \tag{8.38}$$

for every $x \in \mathcal{D}(A)$.

Closed graph theorem $\Rightarrow$ bounded inverse theorem. This part is left as an exercise. (Problem 8.12.)

Bounded inverse theorem $\Rightarrow$ open mapping theorem. We prove this only in the case where X is a Hilbert space. Since A is bounded, $\mathcal{N}(A)$ is closed (cf. Problem 8.9). Thus, we can use the projection theorem to decompose X into $X = \mathcal{N}(A) \oplus \mathcal{N}(A)^\perp$. We then let $P : X \to \mathcal{N}(A)^\perp$ be

Kolmogorov's Calculus of Problems and Solutions

Zur Deutung der intuitionistischen Logik.

Von

A. Kolmogoroff in Moskau.

Die vorliegende Abhandlung kann von zwei ganz verschiedenen Standpunkten aus betrachtet werden.

1. Wenn man die intuitionistischen erkenntnistheoretischen Voraussetzungen nicht anerkennt, so kommt nur der erste Paragraph in Betracht. Die Resultate dieses Paragraphen können etwa wie folgt zusammengefaßt werden:

Neben der theoretischen Logik, welche die Beweisschemata der theoretischen Wahrheiten systematisiert, kann man die Schemata der Lösungen von Aufgaben, z. B. von geometrischen Konstruktionsaufgaben, systematisieren. Dem Prinzip des Syllogismus entsprechend tritt hier z. B. das folgende Prinzip auf: *Wenn wir die Lösung von b auf die Lösung von a und die Lösung von c auf die Lösung von b zurückführen können, so können wir auch die Lösung von c auf die Lösung von a zurückführen.*

Man kann eine entsprechende Symbolik einführen und die formalen Rechenregeln für den symbolischen Aufbau des Systems von solchen Aufgabenlösungsschemata geben. So erhält man neben der theoretischen Logik eine neue *Aufgabenrechnung*. Dabei braucht man keine speziellen erkenntnistheoretischen, z. B. intuitionistischen Voraussetzungen.

Es gilt dann die folgende merkwürdige Tatsache: *Nach der Form fällt die Aufgabenrechnung mit der Brouwerschen, von Herrn Heyting neuerdings formalisierten*[1]*), intuitionistischen Logik zusammen.*

2. Im zweiten Paragraphen wird, unter Anerkennung der allgemeinen intuitionistischen Voraussetzungen, die intuitionistische Logik kritisch untersucht; es wird dabei gezeigt, daß sie durch die Aufgabenrechnung ersetzt werden sollte, denn ihre Objekte sind in Wirklichkeit keine theoretischen Aussagen, sondern vielmehr Aufgaben.

[1]) Heyting, Die formalen Regeln der intuitionistischen Logik, Sitz. d. Pens. Akad. (1930), I, S. 42; II, S. 57; III, S. 158.

A. Kolmogoroff, Zur Deutung der intuitionistischen Logik, *Mathematische Zeitschrift*, Band **35** (1932) S. 58–65

Kolmogorov's Calculus of Problems and Solutions

- Kolmogorov states that intuitionistic logic should be replaced by the calculus of problems, for its objects are in reality not theoretical propositions but rather problems.
- Kolmogorov's calculus of problems and solutions suggests interpretations rather of computability theoretic nature than of proof theoretic nature.
- The Medvedev lattice has been developed as a model for Kolmogorov's calculus of problems (and it turned out to be a model for an intermediate logic, called Jankov's logic).
- We will provide another computability theoretic interpretation of Kolmogorov's calculus of problems and solutions.
- While "problems" have been considered as subsets $A \subseteq \mathbb{N}^{\mathbb{N}}$ in the Medvedev lattice, we will choose a more general interpretation.

Mathematical Problems and Solutions

Definition

A mathematical problem is a partial multi-valued map $f :\subseteq X \rightrightarrows Y$.

The idea is that

- There are a certain sets of potential inputs X and outputs Y.
- The domain $D = \mathrm{dom}(f)$ contains the valid instances of the problem.
- $f(x)$ is the set of solutions of the problem f for instance x.

Definition

A solution of a mathematical problem $f :\subseteq X \rightrightarrows Y$ is a map $s :\subseteq X \rightrightarrows Y$ such that $s(x) \subseteq f(x)$ for all $x \in X$.

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Examples of Mathematical Problems

- The Limit Problem is the mathematical problem

$$\lim :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}, \langle p_0, p_1, ...\rangle \mapsto \lim_{i\to\infty} p_i$$

with $\mathrm{dom}(\lim) := \{(x_i) : (x_i) \text{ is convergent}\}$.

- Martin-Löf Randomness is the mathematical problem $\mathsf{MLR} : 2^{\mathbb{N}} \rightrightarrows 2^{\mathbb{N}}$ with

$$\mathsf{MLR}(x) := \{y \in 2^{\mathbb{N}} : y \text{ is Martin-Löf random relative to } x\}.$$

- The Cohesiveness Problem is the mathematical problem $\mathsf{COH} : (2^{\mathbb{N}})^{\mathbb{N}} \rightrightarrows 2^{\mathbb{N}}$ where $\mathsf{COH}(R_i)$ contains all infinite $X \subseteq \mathbb{N}$ such that for all $i \in \mathbb{N}$ one of the sets

$$X \cap R_i \text{ or } X \cap (\mathbb{N} \setminus R_i)$$

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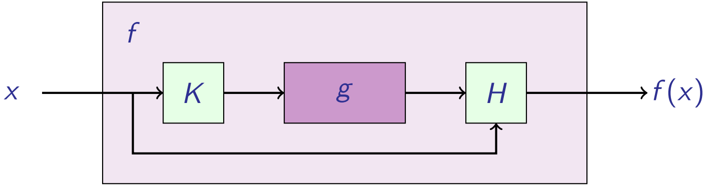
Weihrauch Reducibility

Let $f :\subseteq X \rightrightarrows Y$ and $g :\subseteq Z \rightrightarrows W$ be two mathematical problems.



- f is called Weihrauch reducible to g, in symbols $f \leq_{\mathrm{W}} g$, if there are computable $H :\subseteq X \times W \rightrightarrows Y$ and $K :\subseteq X \rightrightarrows Z$ such that $H(\mathrm{id}, gK) \subseteq f$ and $\mathrm{dom}(f) \subseteq \mathrm{dom}(H(\mathrm{id}, gK))$.
- f is called strongly Weihrauch reducible to g, in symbols $f \leq_{\mathrm{sW}} g$, if there are computable $H :\subseteq W \rightrightarrows Y$ and $K :\subseteq X \rightrightarrows Z$ such that $HgK \subseteq f$ and $\mathrm{dom}(f) \subseteq \mathrm{dom}(HgK)$.

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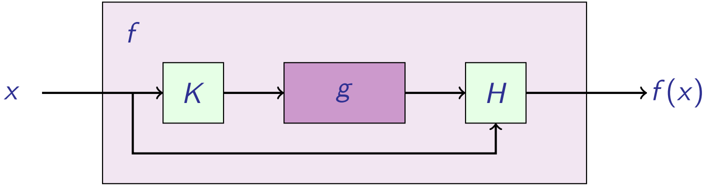

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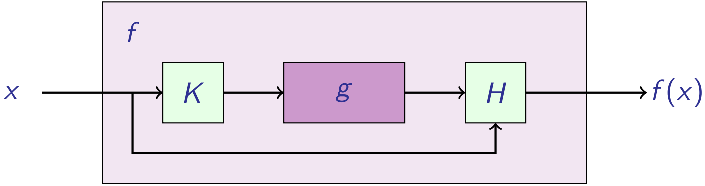

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Algebraic Operations in the Weihrauch Lattice

Definition

Let f, g be two mathematical problems. We consider:

- $f \times g$: both problems are available in parallel (Product)
- $f \sqcup g$: both problems are available, but for each instance one has to choose which one is used (Coproduct)
- $f \sqcap g$: given an instance of f and g, only one of the solutions will be provided (Sum)
- $f * g$: f and g can be used consecutively (Comp. Product)
- $g \to f$: this is the simplest problem h such that f can be reduced to $g * h$ (Implication)
- f^*: f can be used any given finite number of times in parallel (Star)
- $\widehat{f}$: f can be used countably many times in parallel (Parallelization)
- f': f can be used on the limit of the input (Jump)

Some Formal Definitions

Definition

For $f :\subseteq X \rightrightarrows Y$ and $g :\subseteq W \rightrightarrows Z$ we define:

- $f \times g :\subseteq X \times W \rightrightarrows Y \times Z,\ (x,w) \mapsto f(x) \times g(w)$ (Product)
- $f \sqcup g :\subseteq X \sqcup W \rightrightarrows Y \sqcup Z,\ z \mapsto \begin{cases} f(z) \text{ if } z \in X \\ g(z) \text{ if } z \in W \end{cases}$ (Coproduct)
- $f \sqcap g :\subseteq X \times W \rightrightarrows Y \sqcup Z,\ (x,w) \mapsto f(x) \sqcup g(w)$ (Sum)
- $f^* :\subseteq X^* \rightrightarrows Y^*,\ f^* = \bigsqcup_{i=0}^{\infty} f^i$ (Star)
- $\widehat{f} :\subseteq X^{\mathbb{N}} \rightrightarrows Y^{\mathbb{N}},\ \widehat{f} = \mathsf{X}_{i=0}^{\infty} f$ (Parallelization)

- Weihrauch reducibility induces a lattice with the coproduct $\sqcup$ as supremum and the sum $\sqcap$ as infimum.
- Parallelization and star operation are closure operators in the Weihrauch lattice.
- With $\sqcup, \times, ^*$ one obtains a Kleene algebra.
- The Weihrauch lattice is neither a Brouwer nor a Heyting algebra (Higuchi und Pauly 2012).

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Infima and Suprema

The Weihrauch lattice is not complete and infinite suprema and infima do not always exist. There are some known existent ones.

Definition

For two mathematical problem f, g we define the compositional product

$$f * g := \max\{f_0 \circ g_0 : f_0 \leq_{\mathrm{W}} f \text{ and } g_0 \leq_{\mathrm{W}} g\}$$

and the implication

$$g \to f := \min\{h : f \leq_{\mathrm{W}} g * h\}.$$

The maximum and minimum is understood with respect to $\leq_{\mathrm{W}}$ and they always exist (B. and Pauly 2013).

Embedding of the Medvedev Lattice

Definition

Let $A \subseteq \mathbb{N}^\mathbb{N}$.

1. By $c_A : \mathbb{N}^\mathbb{N} \rightrightarrows \mathbb{N}^\mathbb{N}, p \mapsto A$ we denote the constant multi-valued function with value $A \subseteq \mathbb{N}^\mathbb{N}$.
2. By $\mathrm{id}|_A :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ we denote the identity restricted to A.

Proposition (B. and Gherardi 2009)

Let $A, B \subseteq \mathbb{N}^\mathbb{N}$, $A \oplus B = \langle A \times B \rangle$, $A \otimes B = 0A \cup 1B$. Then

- $A \leq_{\mathrm{M}} B \iff c_A \leq_{\mathrm{W}} c_B \iff \mathrm{id}|_B \leq_{\mathrm{W}} \mathrm{id}|_A$,
- $c_{A \oplus B} \equiv_{\mathrm{W}} c_A \times c_B \equiv_{\mathrm{W}} (c_A \sqcup c_B)^* \equiv_{\mathrm{W}} \widehat{c_A \sqcup c_B}$,
- $c_{A \otimes B} \equiv_{\mathrm{W}} c_A \sqcap c_B$,
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Theorems as Problems

Definition

Any theorem T of the form

$$(\forall x \in X)(\exists y \in Y)\,(x \in D \implies P(x,y))$$

is identified with $F :\subseteq X \rightrightarrows Y$ with $\mathrm{dom}(F) := D$ and

$$F(x) := \{y \in Y : P(x,y)\}.$$

Example

Weak Weak Kőnig's Lemma is the mathematical problem

$$\mathsf{WWKL} :\subseteq \mathsf{Tr} \rightrightarrows 2^{\mathbb{N}},\ T \mapsto [T]$$

with $\mathrm{dom}(\mathsf{WWKL}) := \{T \in \mathsf{Tr} : \mu([T]) > 0\}.$

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Choice

Definition

The choice problem C_X of a topological space X is the mathematical problem induced by the statement:

- Every non-empty closed set $A \subseteq X$ has a point $x \in A$.

Example:

- C_2 is the problem of finding a point in a non-empty subset $A \subseteq \{0, 1\}$ where A is described by an infinite sequence that can eventually remove one point from A.
- We obtain $\mathsf{LLPO} \equiv_{\mathrm{sW}} \mathsf{C}_2$ where LLPO is Bishop's Lesser Limited Principle of Omniscience.

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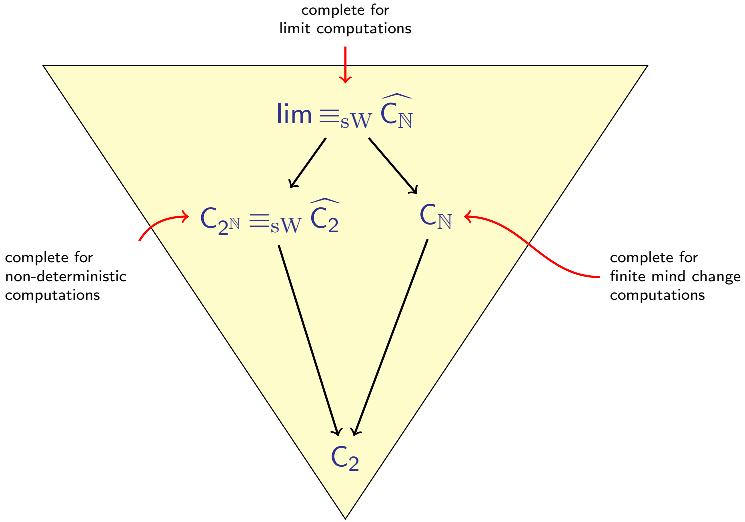
Calibration of Computabiltiy Notions



(Joint results with de Brecht and Pauly 2012)

Discriminative Problems

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We call f discriminative, if $\mathrm{C}_2 \leq_{\mathrm{W}} f$ and indisciminative otherwise.

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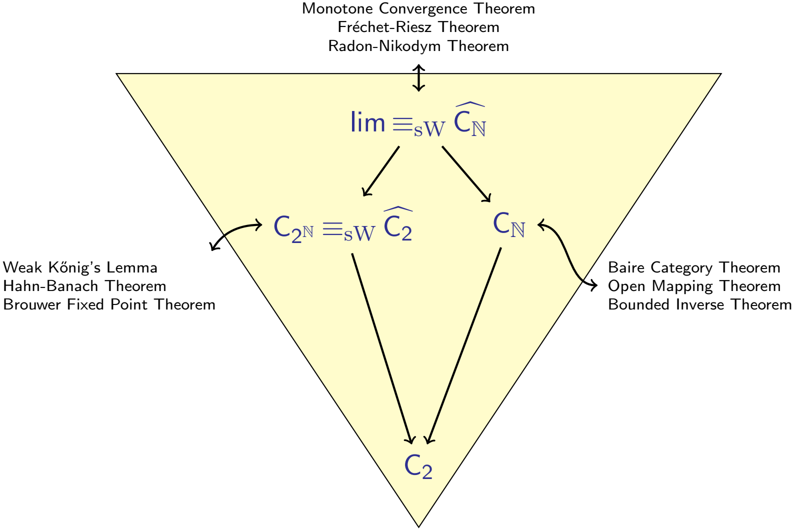
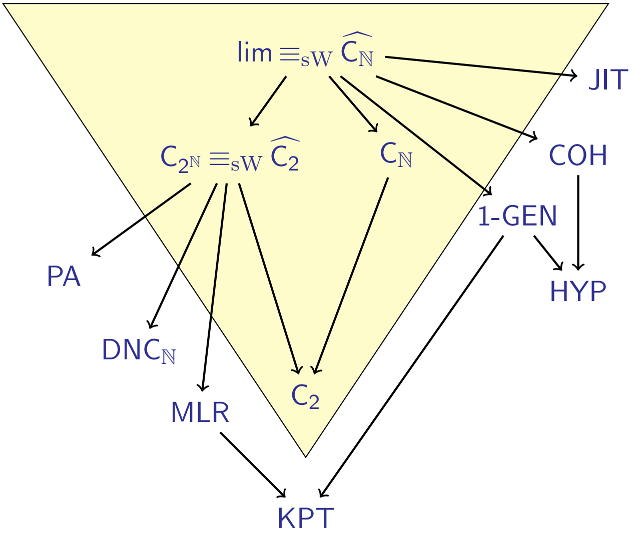
$$\begin{tikzcd}[column sep=small]
 & \lim \equiv_{\mathrm{sW}} \widehat{\mathsf{C}_{\mathbb{N}}} \arrow[dl] \arrow[dr] & \\
\mathsf{C}_{2^{\mathbb{N}}} \equiv_{\mathrm{sW}} \widehat{\mathsf{C}_2} \arrow[dr] & & \mathsf{C}_{\mathbb{N}} \arrow[dl] \\
 & \mathsf{C}_2 &
\end{tikzcd}$$

Classical Analysis is Discriminative



(Results of B., Gherardi, Marcone, Hoyrup, Rojas, Weihrauch, ...)

Classical Computability Theory is Indiscriminative



(Joint results with Hendtlass and Kreuzer)

Classical Computability Theory is Indiscriminative

Philosophical implications:

- Classical mathematics is discriminative.
- Computablity theory is indiscriminative.
- Hence, computability theory has no implications on classical mathematics!

Very general:

- Every theorem that claims the existence of a Turing degree is indiscriminative.

Exceptions in computability theory:

- Theorems such as the Low Basis Theorem are discriminative.

Perhaps that is why it is called the most applicable theorem of computability theory?

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Examples of Implications

Characterization of Martin-Löf Randomness

Theorem (B. and Pauly)

$\mathsf{MLR} \equiv_{\mathrm{W}} (\mathsf{C}_{\mathbb{N}} \to \mathsf{WWKL})$.

Proof. (Sketch.) $(\mathsf{C}_{\mathbb{N}} \to \mathsf{WWKL}) \leq_{\mathrm{W}} \mathsf{MLR}$: It suffices to prove $\mathsf{WWKL} \leq_{\mathrm{W}} \mathsf{C}_{\mathbb{N}} * \mathsf{MLR}$. By Kučera's Lemma, every Martin-Löf random real p is a path in every infinite binary tree T of positive measure up to some finite prefix. Using $\mathsf{C}_{\mathbb{N}}$ we can cut away longer and longer prefixes of p until we find a path in T.

$\mathsf{MLR} \leq_{\mathrm{W}} (\mathsf{C}_{\mathbb{N}} \to \mathsf{WWKL})$: Given some h with $\mathsf{WWKL} \leq_{\mathrm{W}} \mathsf{C}_{\mathbb{N}} * h$ we need to prove that $\mathsf{MLR} \leq_{\mathrm{W}} h$. Given some universal Martin-Löf test $(U_i)_i$, the complement $A_0 := 2^{\mathbb{N}} \setminus U_0$ is a closed set of positive measure and given the corresponding tree T with $A = [T]$ the function h will deliver some sequence q that can be converted into a Martin-Löf random real by a finite mind change computation. This computation can be converted into a regular computation that yields a Martin-Löf random real. $\square$

Characterization of Martin-Löf Randomness

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Proof. (Sketch.) $(\mathsf{C}_{\mathbb{N}} \to \mathsf{WWKL}) \leq_{\mathrm{W}} \mathsf{MLR}$: It suffices to prove $\mathsf{WWKL} \leq_{\mathrm{W}} \mathsf{C}_{\mathbb{N}} * \mathsf{MLR}$. By Kučera's Lemma, every Martin-Löf random real p is a path in every infinite binary tree T of positive measure up to some finite prefix. Using $\mathsf{C}_{\mathbb{N}}$ we can cut away longer and longer prefixes of p until we find a path in T.

$\mathsf{MLR} \leq_{\mathrm{W}} (\mathsf{C}_{\mathbb{N}} \to \mathsf{WWKL})$: Given some h with $\mathsf{WWKL} \leq_{\mathrm{W}} \mathsf{C}_{\mathbb{N}} * h$ we need to prove that $\mathsf{MLR} \leq_{\mathrm{W}} h$. Given some universal Martin-Löf test $(U_i)_i$, the complement $A_0 := 2^{\mathbb{N}} \setminus U_0$ is a closed set of positive measure and given the corresponding tree T with $A = [T]$ the function h will deliver some sequence q that can be converted into a Martin-Löf random real by a finite mind change computation. This computation can be converted into a regular computation that yields a Martin-Löf random real. $\square$

Characterization of Martin-Löf Randomness

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Characterization of Cohesiveness

Theorem (B., Hendtlass and Kreuzer)

$\mathsf{COH} \equiv_{\mathrm{W}} (\lim \to \mathsf{WKL}')$.

Proof. (Idea.) It is known that

- $\mathsf{WKL}' \equiv_{\mathrm{W}} \mathsf{BWT}_{\mathbb{R}}$ (B., Gherardi and Marcone 2012),
- $\mathsf{COH} \equiv_{\mathrm{W}} \mathsf{WBWT}_{\mathbb{R}}$ (Kreuzer 2012),

where $\mathsf{BWT}_{\mathbb{R}}$ is the Bolzano-Weierstraß Theorem and $\mathsf{WBWT}_{\mathbb{R}}$ is the weak Bolzano-Weierstraß Theorem. This implies

$\mathsf{WKL}' \leq_{\mathrm{W}} \lim * \mathsf{COH}$ and hence $(\lim \to \mathsf{WKL}') \leq_{\mathrm{W}} \mathsf{COH}$.
$\mathsf{COH} \leq_{\mathrm{W}} (\lim \to \mathsf{WKL}')$: Given some h with $\mathsf{WKL}' \leq_{\mathrm{W}} \lim * h$ can also prove that $\mathsf{COH} \leq_{\mathrm{W}} h$. □

Theorem (B., Hendtlass and Kreuzer)

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Proof. (Idea) Use a double jump theorem. □

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Research Question

- Find other characterizations of the form

$$f \equiv_{\mathrm{W}} (g \to h).$$

- This is of particular interest when f is indiscriminative and g, h are discriminative.
- In this case it establishes a bridge between the discriminative world and the non-discriminative world since it entails

$$h \leq_{\mathrm{W}} g * f.$$

- Basically, no other characterizations of this form are known, besides the two mentioned ones for MLR and COH.
- An interesting candidate would be, for instance 1-GEN.

Ramsey's Theorem

Ramsey's Theorem

- By $\mathcal{C}_{n,k}$ we denote the set of colorings $c : [\mathbb{N}]^n \to k$.
- By $\mathcal{H}_c$ we denote the set of infinite homogeneous sets for the coloring c.
- A coloring $c : [\mathbb{N}]^n \to k$ is called *stable*, if $\lim_{i\to\infty} c(A \cup \{i\})$ exists for all $A \in [\mathbb{N}]^{n-1}$.
- $\mathsf{RT}_{n,k} : \mathcal{C}_{n,k} \rightrightarrows 2^{\mathbb{N}}, \mathsf{RT}_{n,k}(c) := \mathcal{H}_c$,
- $\mathsf{CRT}_{n,k} : \mathcal{C}_{n,k} \rightrightarrows k \times 2^{\mathbb{N}}, \mathsf{CRT}_{n,k}(c) := \{(c(M), M) : M \in \mathcal{H}_c\}$,
- $\mathsf{SRT}_{n,k} :\subseteq \mathcal{C}_{n,k} \rightrightarrows 2^{\mathbb{N}}, \mathsf{SRT}_{n,k}(c) := \mathsf{RT}_{n,k}(c)$,
  where $\mathrm{dom}(\mathsf{SRT}_{n,k}) := \{c \in \mathcal{C}_{n,k} : c \text{ stable}\}$,

Lower Bounds on Ramsey

Theorem (B., Rakotoniaina)

$\mathsf{C}_2^{(n)} \leq_{\mathrm{W}} \mathsf{RT}_{n,2}$ *for all* $n \geq 1$*.*

Proof.(Idea.) We note that $\mathsf{C}_2^{(n)} \equiv_{\mathrm{sW}} \mathsf{BWT}_2 \circ \lim_{2^{\mathbb{N}}}^{[n-1]}$. Let $p \in \mathrm{dom}(\mathsf{BWT}_2 \circ \lim_{2^{\mathbb{N}}}^{[n-1]})$ and $q := \lim_{2^{\mathbb{N}}}^{[n-1]}(p)$. Then

$$q(i_0) = \lim_{i_1 \to \infty} \lim_{i_2 \to \infty} \dots \lim_{i_{n-1} \to \infty} p\langle i_{n-1}, ..., i_0 \rangle$$

for all $i_0 \in \mathbb{N}$. We compute the coloring $c : [\mathbb{N}]^n \to 2$ with

$$c\{i_0 < i_1 < .... < i_{n-1}\} := p\langle i_{n-1}, i_{n-2}, ..., i_1, i_0 \rangle.$$

For $M \in \mathsf{RT}_{n,2}$ we obtain $c(M) \in \mathsf{BWT}_2(q)$. $\square$

Corollary

$\mathsf{WKL}^{(n)} \leq_{\mathrm{W}} \widehat{\mathsf{RT}_{n,k}}$ *for all* $n \geq 1$, $k \geq 2$*.*

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Products and Parallellization of Ramsey

Theorem (B., Rakotoniaina)

$\mathrm{RT}_{n,\mathbb{N}} \times \mathrm{RT}_{n+1,k} \leq_{\mathrm{sW}} \mathrm{RT}_{n+1,k+1}$ *for all* $n, k \geq 1$.

Proof. (Idea.) Given a coloring $c_1 : [\mathbb{N}]^n \to \mathbb{N}$ with finite range and a coloring $c_2 : [\mathbb{N}]^{n+1} \to k$ we construct a coloring $c^+ : [\mathbb{N}]^{n+1} \to k+1$ as follows:

$$c^+(A) := \begin{cases} c_2(A) & \text{if } A \text{ is homogeneous for } c_1 \\ k & \text{otherwise} \end{cases}$$

for all $A \in [\mathbb{N}]^{n+1}$. Then $\mathrm{RT}_{n+1,2}(c^+) \subseteq \mathrm{RT}_{n,\mathbb{N}}(c_1) \cap \mathrm{RT}_{n+1,k}(c_2)$ and hence the desired reduction follows. $\square$

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$\mathrm{RT}^*_{n,k} \leq_{\mathrm{W}} \mathrm{RT}_{n+1,2}$ *for all* $n, k \geq 1$.

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Parallelization of Ramsey

Theorem (B., Rakotoniaina)

$\widehat{\mathsf{RT}_{n,k}} \leq_{\mathrm{sW}} \mathsf{RT}_{n+2,2}$ *for all* $n, k \geq 1$.

Proof. Given a sequence $(c_i)_i$ of colorings $c_i : [\mathbb{N}]^n \to k$, we compute a sequence $(d_m)_m$ of colorings $d_m \in \mathcal{C}_{n,k^m}$ that capture the products $\mathsf{RT}_{n,k}^m$ and a sequence $(d_m^+)_m$ of colorings $d_m^+ : [\mathbb{N}]^{n+1} \to 2$ by

$$d_m^+(A) := \begin{cases} 0 & \text{if } A \text{ is homogeneous for } d_m \\ 1 & \text{otherwise} \end{cases}$$

for all $A \in [\mathbb{N}]^{n+1}$. Now, in a final step we compute a coloring $c : [\mathbb{N}]^{n+2} \to 2$ with $c(\{m\} \cup A) := d_m^+(A)$ for all $A \in [\mathbb{N}]^{n+1}$ and $m < \min(A)$. Given an infinite homogeneous set $M \in \mathsf{RT}_{n+2,2}(c)$ we determine a sequence $(M_i)_i$ as follows: for each fixed $i \in \mathbb{N}$ we first search for a number $m > i$ in M and then we let $M_i := \{x \in M : x > m\}$. $\square$

Lower Bounds

Corollary (B., Rakotoniaina)

For all $n \geq 2$ *we obtain:*

- $\lim \leq_{\mathrm{W}} \mathsf{SRT}_{3,2}$,
- $\mathsf{WKL}' \leq_{\mathrm{W}} \mathsf{RT}_{3,2}$, *(Hirschfeldt and Jockusch 2015)*
- $\mathsf{WKL}^{(n)} \leq_{\mathrm{W}} \mathsf{SRT}_{n+2,2}$.

Upper Bounds

Theorem (Cholak, Jockusch, Slaman 2009)

$\mathsf{RT}_{n,k} \leq_{\mathrm{W}} \mathsf{SRT}_{n,k} * \mathsf{COH}$ *for all* $n, k \geq 1$.

Theorem (B., Rakotoniaina)

$\mathsf{SRT}_{n+1,k} \leq_{\mathrm{W}} \mathsf{RT}_{n,k} * \lim$ *for all* $n, k \geq 1$.

Proof. (Idea.) If fact, we even proved $\mathsf{CRT}'_{n,k} \equiv_{\mathrm{W}} \mathsf{SRT}_{n+1,k}$. □

Corollary (B., Rakotoniaina)

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Parallelization and Borel Complexity

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Corollary

$\mathsf{RT}_{n,k}$ *is effectively* $\boldsymbol{\Sigma}^0_{n+2}$*–measurable, but not* $\boldsymbol{\Sigma}^0_{n+1}$*–measurable for all* $n, k \geq 2$.

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Color Separation

Theorem (Squashing Theorem)

$f \times g \leq_{\mathrm{W}} g \Longrightarrow \widehat{f} \leq_{\mathrm{W}} g$ *for total* f, g *and finitely tolerant* f.

Corollary (Dorais, Dzhafarov, Hirst, Mileti and Shafer)

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$f \times g \leq_{\mathrm{W}} g \Longrightarrow \widehat{f} \leq_{\mathrm{W}} g$ *for total* f, g *and finitely tolerant* f.

Corollary (Dorais, Dzhafarov, Hirst, Mileti and Shafer)

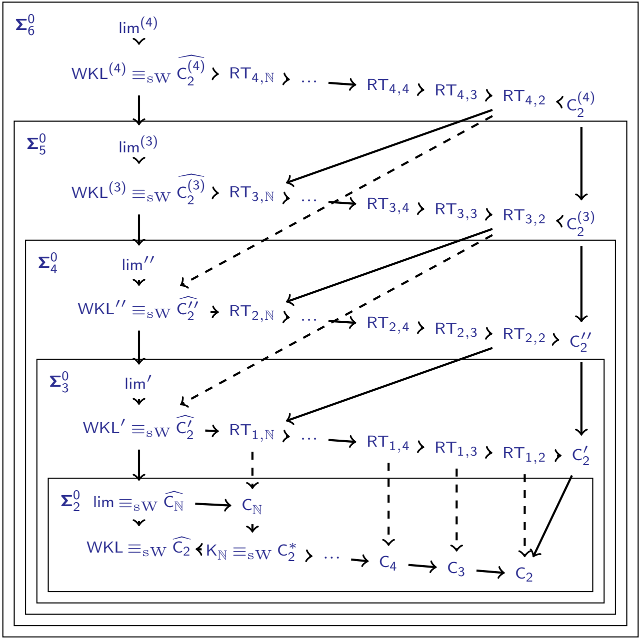
$\mathsf{RT}_{n,k} <_{\mathrm{sW}} \mathsf{RT}_{n,k+1}$ *for all* $n, k \geq 1$.

Theorem (B. & Rakotoniaina, Hirschfeldt & Jockusch, Patey)

$\mathsf{RT}_{n,k} <_{\mathrm{W}} \mathsf{RT}_{n,k+1}$ *for all* $n, k \geq 1$.

Proof. The assumption $\mathsf{RT}_{n,2} \times \mathsf{RT}_{n+1,k} \leq_{\mathrm{W}} \mathsf{RT}_{n+1,k}$ implies by the Theorem $\widehat{\mathsf{RT}_{n,2}} \leq_{\mathrm{W}} \mathsf{RT}_{n+1,k}$ and hence by our lower bound $\lim^{(n-1)} \leq_{\mathrm{W}} \mathsf{WKL}^{(n)} \equiv_{\mathrm{W}} \widehat{\mathsf{RT}_{n,2}} \leq_{\mathrm{W}} \mathsf{RT}_{n+1,k}$ in contradiction to the Cone Avoidance Theorem of Cholak, Jockusch and Slaman. Hence $\mathsf{RT}_{n,2} \times \mathsf{RT}_{n+1,k} \not\leq_{\mathrm{W}} \mathsf{RT}_{n+1,k}$ for all $n, k \geq 1$. However $\mathsf{RT}_{n,2} \times \mathsf{RT}_{n+1,k} \leq_{\mathrm{W}} \mathsf{RT}_{n+1,k+1}$ by our Product Theorem, i.e., $\mathsf{RT}_{n+1,k} <_{\mathrm{W}} \mathsf{RT}_{n+1,k+1}$ for all $n, k \geq 1$. $\square$

Ramsey's Theorem in the Weihrauch Lattice



Conclusion

1. The purpose of this talk was to demonstrate in different case studies how algebraic properties in the Weihrauch can be applied to prove interesting characterizations.
2. Often such characterizations boil down to an identification of the right algebraic properties of the problems involved.
3. This often leads to very transparent and simple proofs of properties that are otherwise hard to obtain.

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