Generic, Coarse, Cofinite, and Mod-Finite Reducibilities

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In complexity theory, it has been observed that problems can be difficult in theory while being quite easy to solve in practice.

1986: Levin introduces "average-case complexity."

2003: Kapovich, Miasnikov, Schupp and Shpilrain introduce "generic-case complexity."

Density

In 2012, Jockusch, and Schupp introduce and analyze the notion of generic computability. Informally, real is genericaly computable if there is a computation of that real that is usually correct.

We formalize our notion of "usually" using asymptotic density:

Definition

The density of real A is the limit of the densities of its initial segments, $\lim_{n\to\infty} \frac{|A\cap n|}{n}$.

Generic and coarse computablity

Definition

A real A is generically computable if there exists a partial computable function ϕ whose domain has density 1 such that $\phi(n) = A(n)$ for all $n \in \mathrm{dom}(\phi)$.

Definition

A real A is coarsely computable if there exists a total computable function ϕ such that $\{n : \phi(n) = A(n)\}$ has density 1.

So a generic computation is a computation that usually halts, always correctly, while a coarse computation is a computation that always halts, usually correctly.

Partial Oracles

Definition

Let A be a real. Then a (time-dependent) partial oracle, (A), for A is a set of ordered triples $\langle n, x, s\rangle$ such that:
$\exists s\big(\langle n, 0, s\rangle \in (A)\big) \implies n \notin A$,
$\exists s\big(\langle n, 1, s\rangle \in (A)\big) \implies n \in A$.

We think of (A) as a partial function, sending n to x. We think of s as the number of steps it takes (A) to converge.

The domain of (A) is the set of n for which there exists such an x, s.

Generic reduction

Definition

Let A be a real. Then a generic oracle for A is a partial oracle whose domain is density-1.

Note that generically computing A is equivalent to computing a generic oracle for A.

Definition

Let A, B be reals. We say A is (uniformly) generically reducible to B (or $A \leq_{\text{gen}} B$) if there is a Turing functional ϕ such that for every generic oracle (B), for B, $\phi^{(B)}$ is a generic computation of A.

In nonuniform generic reduction, the choice of ϕ is allowed to depend on (B).

Coarse reduction

Definition

Let A be a real. Then a coarse oracle for A is an (ordinary Turing) oracle for a set that agrees with A on density-1.

Definition

Let A, B be reals. We say A is coarsely reducible to B (or $A \leq_{\text{cor}} B$) if there is a Turing functional ϕ such that for every coarse oracle C, for B, ϕ^C is a coarse computation of A.

In nonuniform coarse reduction, the choice of ϕ is allowed to depend on C.

Observations

Fact

Uniform generic reduction and nonuniform coarse reduction seem to be easier to work with than nonuniform generic reduction or uniform coarse reduction.

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Embedding the Turing degrees in the generic degrees

There is a natural embedding of the Turing degrees into the generic degrees:

Definition

For any real X, let $\mathcal{R}(X)$ be defined as follows.
$\mathcal{R}(X) = \{2^n(2k+1) : n \in X\}$.

So we have "stretched" every bit of X into a positive density "column" of $\mathcal{R}(X)$.

Since every generic oracle for $\mathcal{R}(X)$ must include at least one bit from every column, it must be able to compute X.

As a result, generically computing $\mathcal{R}(X)$ is the same as computing X, and working with $\mathcal{R}(X)$ as a generic oracle is the same as working with X as an oracle in the usual sense.

Note that this embedding fails quite badly for the coarse degrees.

Observation

If A is Δ^0_2, then $\mathcal{R}(A)$ is coarsely computable.

What if we had done something differently?

Definition

For any real X, let $\mathcal{I}(X)$ be defined as follows.
$\mathcal{I}(X) = \bigcup_{n \in X} [n!, (n+1)!)$.

In this case, we have "stretched" each bit of X over a larger and larger finite initial segment of $\mathcal{I}(X)$.

A generic oracle for $\mathcal{I}(X)$ can "miss" finitely many bits of X, but from some point on must have all of them.

Each bit is coded in a finite number of locations, so a coarse oracle can take a poll. It might make finitely many mistakes about X, but from some point on, it will be correct.

Cofinite and Mod-Finite reducibilities

Definition

Let A, B be reals. We say A is cofinitely reducible to B (or $A \leq_{\text{cf}} B$) if there is a Turing functional ϕ such that for every partial oracle (B), for B, if (B) has cofinite domain, then $\phi^{(B)}$ is a partial computation of A with cofinite domain.

Definition

Let A, B be reals. We say A is mod-finitely reducible to B (or $A \leq_{\text{mf}} B$) if there is a Turing functional ϕ C, if $C =^* B$, then ϕ^C is a computation of a set that is mod-finitely equal to A.

Embeddings

Theorem (Dzhafarov, I.)

$$A \leq_{cf} B \iff \mathcal{I}(A) \leq_{gen} \mathcal{I}(B)$$
$$A \leq_{mf} B \iff \mathcal{I}(A) \leq_{cor} \mathcal{I}(B)$$

Theorem (Dzhafarov, I.)

$$A \leq_T B \iff \mathcal{R}(A) \leq_{gen} \mathcal{R}(B)$$
$$A \leq_T B \iff \mathcal{R}(A) \leq_{cf} \mathcal{R}(B)$$
$$A \leq_T B \iff \mathcal{R}(A) \leq_{mf} \mathcal{R}(B)$$

Theorem (Dzhafarov, I.; Hirschfeldt, Jockusch, Kuyper, Schupp)

$$A \leq_T B \iff \mathcal{I}(\mathcal{R}(A)) \leq_{gen} \mathcal{I}(\mathcal{R}(B)) \qquad \textit{(DI)}$$
$$A \leq_T B \iff \mathcal{I}(\mathcal{R}(A)) \leq_{cor} \mathcal{I}(\mathcal{R}(B)) \qquad \textit{(DI;HJKS)}$$

Theorem (Dzhafarov, I.)

$(A \leq_1 B) \Rightarrow (A \leq_{mf} B) \Rightarrow (A \leq_{cf} B) \Rightarrow (A \leq_T B)$.
All these implications are strict.

For our purposes, the center implication is the interesting implication. There is no implication between $\leq_{\text{gen}}$ and $\leq_{\text{cor}}$.

It uses a "guessing trick":
We have that $A \leq_{\text{mf}} B$ via ϕ.
By finite modification of ϕ, we may assume that $\phi^B = A$.
Given a cofinite oracle for B, we guess at every possible value for the bits that we do not yet have. If every guess gives the same output when used as an oracle, then we halt and give that output.
Compactness ensures that our domain is cofinite.

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- Questions and Results

Minimal degrees and pairs

Question

Assume A, is not generically computable. Is there a density-1 real B such that $0 <_{gen} B \leq_g A$*?*

If the answer to the question is “yes,” then there cannot be any minimal generic degrees, because the density-1 degrees are dense. (I.)

If the answer to the question is “no,” then the counterexample is half of a minimal pair for generic reduction. (I.)

Also, the generic degrees of density-1 reals are exactly the generic degrees of coarsely computable reals.

Two more definitions

Definition

In the generic degrees, A is density-1 bounding if there is a density-1 real B such that $0 <_{\text{gen}} B \leq_g A$.

Definition

In the generic degrees A is quasiminimal if there is a noncomputable B such that $\mathcal{R}(B) \leq_g A$.

In fact, in any of our degree structures, we say that A is quasi-minimal if its degree lies above a nonzero embedded Turing degree.

Theorem (I.)

In the generic degrees, if A is not quasi-minimal, then A is density-1 bounding.

So if we wish to attempt to build a real that is not density-1 bounding, it must be through a construction that is capable of building sets that are not quasi-minimal.

So how does one build quasi-minimal sets?

Theorem (Jockusch, Schupp)

There is a quasi-minimal set.

Their set is density-1.

Theorem (I.)

For any noncomputable A, B, there is a non-generically-computable C such that $C \leq_g \mathcal{R}(A)$, and $C \leq_g \mathcal{R}(B)$.

Note: If A and B are a minimal pair in the Turing degrees, then C must be quasi-minimal. The constructed C is density-1.

Observation

If A is 1-generic or 1-random, then A is density-1 bounding.

Theorem (I.)

If A is noncomputable, then $\mathcal{I}(A)$ is density-1 bounding.

Theorem (Cholak, I.)

If A is 1-generic or 1-random, then A is quasi-minimal in the generic degrees. In fact, A is quasi-minimal in the cofinite degrees.

Big Lemma (Cholak, I.)

If A is quasi-minimal in the cofinite degrees, then it is quasi-minimal in the mod-finite, generic, and coarse degrees.

Theorem (Hirschfeldt, Jockusch, Kuyper, Schupp)

In the nonuniform coarse degrees, and therefore the uniform coarse degrees, weakly 2-randoms are quasi-minimal, but there exist 1-randoms that are not quasi-minimal.

Proof of Big Lemma

- If $\mathcal{R}(B) \leq_{\text{mf}} A$, then $\mathcal{R}(B) \leq_{\text{cf}} A$ because of the implication between cofinite and mod-finite reducibility.

  So B must be computable.
- If $\mathcal{I}(\mathcal{R}(B)) \leq_{\text{gen,cor}} A$, then $\mathcal{I}(\mathcal{R}(B)) \leq_{\text{gen,cor}} \mathcal{R}(A)$ because $A \leq_{\text{gen,cor}} \mathcal{R}(A)$.

  But in that case, we have that $\mathcal{R}(B) \leq_{\text{cf,mf}} A$ because $\mathcal{I}$ embeds the Turing degrees into both the co-finite degrees and the mod-finite degrees.

  So, again, B must be computable.

Carryovers

The Big Lemma allows us carry over the proof that 1-randoms are quasi-minimal in the uniform generic degrees to show that they are also quasi-minimal in the uniform coarse degrees.

A fairly short modification of the proof from HJKS allows them to show that there exist 1-randoms that are not quasi-minimal in the nonuniform generic degrees.

Question

Are weakly 2-randoms quasi-minimal in the nonuniform generic degrees?

End

Thank you for your attention.

References

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