The Randomness of Empirical Data, the Simplicity of Hypotheses, and Extreme Priors

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Overview

Two Philosophical Uses of Computability

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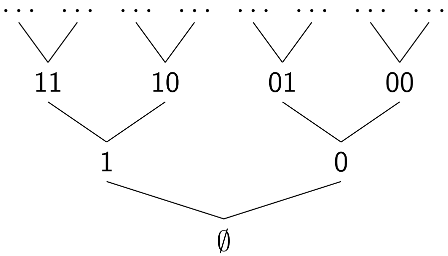
- Isolated attempts to apply computability theory to the scientific endeavor
- This project is a preliminary effort to connect these various attempts

Scientific Hypotheses and Cantor Space

- We represent scientific hypotheses as subsets of Cantor space (henceforth 2^ω)
- Recall: 2^ω is the set of all functions $f : \omega \to \{0, 1\}$



- We code streams of data as individual sequences of 0's and 1's
- A hypothesis is the collection of all data streams that would make the hypothesis true

The Topology of Cantor Space

- Recall the standard topology on Cantor space
- For any finite sequence s, the open fan $[s] = \{x \in 2^{\omega} | x \restriction ln(s) = s\}$
- The open sets of 2^{ω} are all arbitrary unions of fans
- The closed sets are the complements of unions of fans

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- The open sets of 2^ω are all arbitrary unions of fans
- The closed sets are the complements of unions of fans
- Ex: As $\{x \in 2^\omega | \exists n \in \mathbb{N}(x(n) = 1)\} = \bigcup_{n<\omega}[0^n 1]$, it is open
  - This set corresponds to the claim that a 1 will eventually be observed
- Ex: As $\{\bar{0}\} = 2^\omega - \{x \in 2^\omega | \exists n \in \mathbb{N} x(n) = 1\}$, it is closed.
  - This set corresponds to the claim that a 1 will never be observed

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- It follows that null sets will be extremely common

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- Standard Bayesian conditionalization allows changes to the calculated probabilities of most hypotheses through inquiry
- Sets with extreme priors, where $\mu(X) = 0$ or $\mu(X) = 1$, however will be immune to changes by conditionalization
  - Thus, the collection of null sets will remain unchanged throughout inquiry

The Philosophical Problem: The Limitations of Inquiry

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  - Ex: No matter how many days the sun is seen to rise, we cannot be certain that the sun will rise every day.
- Duhem's Thesis notes that hypotheses are often tested in groups
- When a collection of hypotheses makes a false prediction, any one of them can be rejected as the problematic assumption
  - Ex: Ptolemaic astronomy can respond to false predictions by rejecting simple planetary orbits, instead of rejecting that planets orbit the sun

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- There are degress of underdetermination: "all swans are white" is less underdetermined than "all swans (except finitely many) are white"

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- Kelly's Logic of Reliable Inquiry attempts to discover how successful the best method can be on hypotheses of certain levels of complexity

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  - For $\{\bar{0}\}$, conclude it is false only when data of the form $0^n 1$ is observed for finite n
- Similar notions of reliability correspond to levels higher in the Borel hierarchy; for example, F_σ hypotheses will be verifiable-in-the-limit, while G_δ hypotheses will be refutable-in-the-limit

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- The Problem of Induction occurs when a hypothesis can never be verified with certainty
  - This corresponds to a hypothesis being a non-closed set
- A central idea of Kelly's approach: underdetermination of a hypothesis corresponds to the complexity of the hypothesis
- "All swans are white" is topologically less complex than "all swans (except finitely many) are white"; this explains its lower degree of underdetermination

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- We then recursively define the hierarchy as follows:
    - A set is Π^0_n if it is the complement of a Σ^0_n set
    - A set is Σ^0_{n+1} if it is definable by some $\varphi \equiv \exists x \psi$ where ψ is Π^0_n
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- By classical results, the class of Σ^0_1 sets is a subset of the open sets on a topological space
- Similarly, the class of Π^0_1 sets is a subset of the closed sets on a topological space
- Thus, a Σ^0_1 hypothesis is verifiable, and a Π^0_1 hypothesis is refutable (by Kelly's definition)

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Theorem

For any Π^0_1 null class P, there is an ML-test $(G_m)_{m\in\mathbb{N}}$ such that $P = \bigcap_{m\in\mathbb{N}} G_m$; thus, no MLR sequence can be in a Π^0_1 null class

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- Notions of algorithmic randomness, defining random infinite sequences of Cantor space, fit well with Kelly's approach to inquiry
- We will therefore explore adding assumptions that the streams of data are random to our model
- As MLR is a particularly well-behaved notion of randomness, we shall start by considering the assumption that data is MLR

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  - Approaches to scientific inquiry that ignore randomness therefore risk missing a crucial feature
- A combination of these two programs thus seems initially promising

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  - Correctness: H correctly holds of the actual world
- Each of these assumptions seems independently plausible
- Furthermore, there is little reason to expect they could not be jointly satisfied

A Puzzle of Co-Impossibility

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Proof.

By the refutability assumption, H is a Π^0_1 set.
By the nullity assumption, $\mu(H) = 0$.
Thus, H is Π^0_1 null class.
By the randomness assumption, $x \in MLR$.
But, by the earlier stated theorem, $x \in MLR$ implies $x \notin P$ for any Π^0_1 null class P.
Thus, $x \notin H$, and so H cannot hold.
We conclude that the correctness assumption must be false. $\square$

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  one can conclude without empirical investigation that the hypothesis is false
- Such a claim seems to misrepresent the nature of hypothesis testing and the capabilities of human investigators

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- Both Kelly's approach to modeling scientific inquiry and the claim that data is random seemed quite fruitful
- With the goal of generating a more comprehensive understanding of hypothesis testing through empirical data, we set out to combine the approaches
- But a simple combination of the assumptions of the approaches leads to an immediate seeming absurdity
- Thus, one seems forced to alter some aspect(s) of one or both approaches to allow a more satisfactory combination

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- For the remainder of this talk, we will explore both responses

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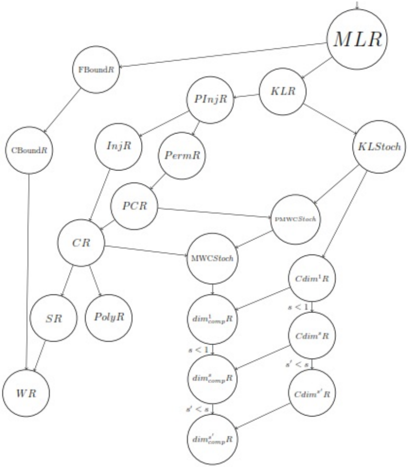
- As $l \to \infty$, we thus find that the maximum compression of initial sequences of data through scientific laws goes to 0
- This characterizes scientific explanation as wholly ineffective.
- So perhaps a notion of randomness with a less severe incompressibility property would be preferable

The Zoo of Randomness Notions

- Furthermore, there is a wide variety of alternative notions of algorithmic randomness[1]



[1] Image credit to Antoine Taveneaux

Randomness and Π^0_1 Null Classes

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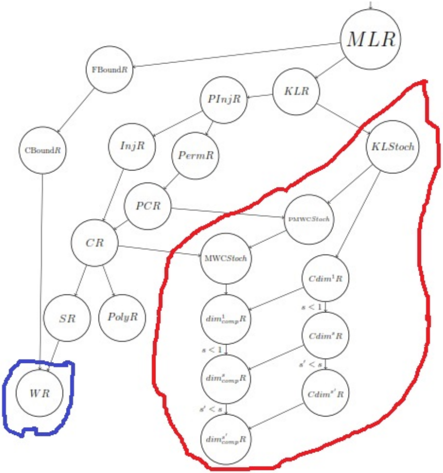
- Additionally, this approach to the puzzle aims to find a formal explication of randomness that is faithful to the pre-theoretic notion
- A particularly low-strength notion of randomness, namely Weak Randomness (*WR*), is characterized by avoiding Π^0_1 classes
- So any acceptable alternative to *MLR* must fail to imply *WR*

A More Restricted Zoo of Randomness Notions

- The need to avoid *WR* actually poses a strong constraint on the notions of randomness that are acceptable



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- Different notions of stochasticity identify the relevant sense of computability in different ways
- Kolmogorov-Loveland Stochasticity (*KLS*) considers all subsequences generated by a decidable, non-monotonic function on the data sequence
- Formally, a set X is *KLS* if no computable selection function has as its range a subset of X with disproportionate numbers of 0's and 1's

Benefits of *KLS*

Theorem (Wang 1996)

$KLS \nrightarrow WR$ and $WR \nrightarrow KLS$

- *KLS* is thus formally capable of solving the puzzle

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- Additionally, *KLS* meets Twardy et al's challenge regarding the proper extent of incompressibility
- As our solution would require a non-*WR* data sequence, there will be no finite limit on the data's maximum compression
- If a set X is KLS, then for any c there will be infinitely many lengths n such that the shortest description of X will be greater than $c * log(n)$
    - Thus, we retain McAllister's intuition that empirical data should never be wholly compressible to a finite scientific law

KLS and Fair Coin Tosses

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  - In particular, Kollektivs characterizes as random a sequence that at every point have a relative frequency of 1's greater than $\frac{1}{2}$
  - Yet "No one would be satisfied with such a sequence as an idealization of actual coin tossing... we should certainly declare the coin to be biased." (Martin-Löf, 1969)

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Theorem (Merkle 2003)

There is a KLS set X such that for all $l \in \mathbb{N}$, $X \upharpoonright l$ has a higher frequency of 1's than 0's

- We thus find that *KLS* is susceptible to the very problem that motivated its creation
- While *KLS* is proposed as a notion of randomness for its ability to correctly distinguish fair and unfair coin tosses, it is unsuccessful in this capacity

Motivating *KLS* as a Notion of Randomness

- *KLS* is defended as a notion of algorithmic randomness wholly on the basis of its connection to the statistical concept of fair coin tosses
  - Resolving Martin-Löf's critique of Kollektivs simply is the motivation for developing *KLS*

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- It is unclear that *KLS* functions as a notion of randomness at all
- While it is technically capable of solving the puzzle, we conclude that *KLS* is nonetheless not able to satisfactorily formalize the claim that data is random

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    - One promising avenue of future study is notions of pseudorandomness from complexity theory.

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- One would then seek to mitigate the unintuitive consequences of the puzzle
- We will focus on the nullity assumption
- Two distinct ways forward might be to limit the ubiquity of null hypotheses, or find compelling argument that these hypotheses can be ignored in a model of hypothesis testing

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- But by Hájek's result, there must be uncountably many null hypotheses
- Thus, regularity cannot be reasonably required of measures in the current context

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- A particularly salient alternative is presented by Rényi, who takes conditional probabilities as primitive
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- However, once data is fixed, as it must be in a model of hypothesis testing, Rényi's second axiom dictates that the alternative approach collapses into the standard account

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- Let $\mathcal{C}$ be a countable collection of Borel subsets of Cantor space. Then say that a probability measure μ is $\mathcal{C}$-*regular* if it satisfies the following condition:

$$(\mathcal{X} \in \mathcal{C} \;\&\; \mathcal{X} \neq \emptyset) \Longrightarrow \mu(\mathcal{X}) > 0 \tag{1}$$

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- Let $\mathcal{C}$ include some significant extent of the Π^0_1 classes
- Perhaps we could restrict attention to only measures that are $\mathcal{C}$-regular
  - Then the scope of the Π^0_1 null hypotheses would be significantly limited, effectively

μ-ML Randomness

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Let μ be a computable measure on 2^ω. Then there is a universal test $(\widehat{U_n})_{n\in\mathbb{N}}$ such that for all x, $x \in MLR_\mu$ if and only if $x \notin \bigcap_{n\in\mathbb{N}} \widehat{U_n}$

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- This provides strong preliminary support for the feasibility of MLR_μ, provided that a computable $\mathcal{C}$-regular measure μ is itself plausible
- It bears further exploration on whether computability and $\mathcal{C}$-regularity requirements, for some meaningful $\mathcal{C}$, can be justified

In Conclusion

- We found a puzzling incompatibility between Kelly's approach to modeling inquiry and intuitive assumptions of the randomness of empirical data
- This incompatibility can be expressed as the co-impossibility of four assumptions: refutability, nullity, randomness, and correctness

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- We found a puzzling incompatibility between Kelly's approach to modeling inquiry and intuitive assumptions of the randomness of empirical data
- This incompatibility can be expressed as the co-impossibility of four assumptions: refutability, nullity, randomness, and correctness
- A technically and philosophically satisfying solution to the puzzle is not readily forthcoming
- Either potential route to a solution requires sustained mathematical and philosophical exploration of less well-studied notions of algorithmic randomness

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