Algorithmically Random Functions and Effective Capacities

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Goals

1. Continue the analysis of algorithmic randomness for closed sets and continuous functions on 2^ω and their connection with Choquet capacity.

2. Study online random functions and partial random functions.

History

Barmpalias, Brodhead, Cenzer et al (JLC 2007, AML 2008, JLC 2009) have developed the notion of algorithmic randomness for closed sets and continuous functions on 2^ω as part of the broad program of algorithmic randomness.

The study of random closed sets was furthered by Axon (2010 Notre Dame Ph.D. thesis), Diamondstone and Kjos-Hanssen (APAL 2012), and others. Cenzer et al (LMCS 2011) studied the relationship between notions of random closed sets with respect to different computable probability measures and effective capacities.

Algorithmic randomness for reals has been extensively studied in recent years. See Downey-Hirschfeldt (2011) and Nies (2009) for more.

Outline

- Background on Algorithmic Randomness and Capacity
- Extending randomness for continuous functions to Bernoulli measures; partial random functions
- From functions to capacities
- Random online functions
- Integrals of random functions

Algorithmically Random Reals

Algorithmic randomness is concerned with the randomness of a single real $x = (x(0), x(1), \dots) \in 2^\omega$ (or in Σ^ω where Σ is finite).

There are characterizations in terms of Kolmogorov Complexity, in terms of martingales or betting stategies, and in terms of statistical tests.

Notions of Randomness for Computable Measures

Space $\mathcal{X} = \Sigma^{\mathbb{N}}$ for some finite Σ ($\{0,1\}$)

Computable measure μ on $\mathcal{X}$, Lebesgue measure λ or biased (Bernoulli) coin toss λ_p – probability p of a 1 and probability $1-p$ of a 0.

More general continuous measures will also be considered.

A μ-Martin-Löf test is an effective sequence $(U_n)_{n\in\omega}$ of c.e. open sets with $\mu(U_n) < 2^{-n}$ for all n.

Thus the intersection $\bigcap_n U_n$ is an effectively defined set having measure zero.

x passes such a test if $x \notin \bigcap_n U_n$.

x is μ-Martin-Löf random if it passes every μ-Martin-Löf test.

Trees and Closed Sets

A subset T of Σ^* is a *tree* if $\tau \in T$ and $\sigma \sqsubset \tau$ implies $\sigma \in T$.

$[T] = \{x \in \Sigma^{\mathbb{N}} : (\forall n) x{\upharpoonright}n \in T\}$.

The intervals $[\sigma]$ for $\sigma \in \Sigma^*$ form a basis for the topology.

Fact: A subset Q of $\Sigma^{\mathbb{N}}$ is closed IFF $Q = [T]$ for some tree T

Q is *effectively closed* IFF $Q = [T]$ for some computable T.

The complement of an effectively closed set is *c.e. open*.

For a closed set Q, let $T_Q = \{\sigma : Q \cap [\sigma] \neq \emptyset$; T_Q is a tree.

$Q = [T_Q]$ for any closed set Q.

Random Closed Sets

Given a closed set $Q = [T]$ where T has no dead ends, let $\sigma_0, \sigma_1, \ldots$ enumerate the elements of T in length-lex order.

The code $x = x_Q$ for Q is defined by recursion such that for each n,

(i) $x(n) = 2$ if both $\sigma_n{}^\frown 0$ and $\sigma_n{}^\frown 1$ are in T,
(ii) $x(n) = 1$ if $\sigma_n{}^\frown 0 \notin T$ and $\sigma_n{}^\frown 1 \in T$, and
(iii) $x(n) = 0$ if $\sigma_n{}^\frown 0 \in T$ and $\sigma_n{}^\frown 1 \notin T$.

We then define a measure μ^* on $\mathcal{C}$ by setting

$$\mu^*(\mathcal{X}) = \mu(\{x_Q : Q \in \mathcal{X}\})$$

for any $\mathcal{X} \subseteq \mathcal{C}$ where μ is any measure on $\{0, 1, 2\}^{\mathbb{N}}$.

Previous Results for Random Closed Sets

- Δ^0_2 random closed sets exist.
- There are no random Π^0_1 closed sets.
- Any random closed set is perfect.
- Any random closed set has measure 0.
- Any random closed set has dimension $\log_2 \frac{4}{3}$.
- Random closed sets contain no n-c.e. elements.

Continuous Functions

A continuous function $F : 2^\omega \to 2^\omega$ may be represented by a function $f : \{0,1\}^* \to \{0,1\}^*$ such that the following hold for all $\sigma \in \{0,1\}^*$.

(1) $|f(\sigma)| \leq |\sigma|$;

(2) $\sigma_1 \sqsubset \sigma_2$ implies $f(\sigma_1) \sqsubseteq f(\sigma_2)$;

(3) For every n, there exists m such that for all $\sigma \in \{0,1\}^m$, $|f(\sigma)| \geq n$;

(4) For all $x \in 2^\omega$, $F(x) = \bigcup_n f(x \lceil n)$.

Coding with delay

Enumerate $\{0,1\}^*$ in length-lex order as $\sigma_0 = \emptyset, \sigma_1 = (0), \sigma_2 = (1), \sigma_3 = (00), \ldots$.

Let $r \in 3^\omega$ correspond to the function $f_r : \{0,1\}^* \to \{0,1\}^*$ defined by declaring that $f_r(\emptyset) = \emptyset$ and that, for any $n > 0$,

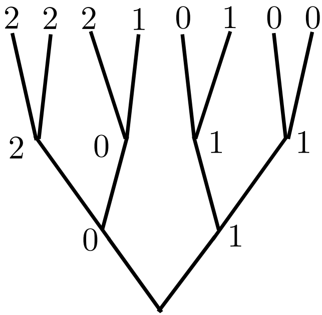
$$f_r(\sigma_n) = \begin{cases} f_r(\sigma_k), & \text{if } r(n) = 2 \\ f_r(\sigma_k)^\frown i, & \text{if } r(n) = i < 2 \end{cases}$$

where k is such that $\sigma_n = \sigma_k{}^\frown j$ for some j.

Then every continuous function F has a representative f as described above, and, in fact, it has infinitely many representations.

For the uniform $1/3$ measure μ, almost every $r \in 3^\omega$ codes a total, continuous function.

$$z = 01201122210100\ldots$$



$$\begin{aligned} 000 &\mapsto 0 & \qquad 100 &\mapsto 110 \\ 001 &\mapsto 0 & 101 &\mapsto 111 \\ 010 &\mapsto 00 & 110 &\mapsto 110 \\ 011 &\mapsto 001 & 111 &\mapsto 110 \end{aligned}$$

Random Functions

A continuous function $F : 2^\omega \to 2^\omega$ is μ^{**}-random if there is a code for F which is μ-random.

Some results from previous work:

- There are random Δ^0_2 continuous functions, but no computable function can be random and no random function can map a computable real to a computable real.
- The image of a random function is a perfect set
- For any $y \in 2^\omega$, there exists a random continuous function F with y in the image of F; thus the image of a random continuous function need not be a random closed set.
- The set of zeroes of a random continuous function is a random closed set (if nonempty).

Symmetric Bernoulli Measures

A measure μ on 3^ω is a *Bernoulli* measure if there are $p_0, p_1, p_2 \in [0,1]$ such that $p_0 + p_1 + p_2 = 1$ and $\mu(\sigma^\frown i) = p_i \cdot \mu(\sigma)$ for each $i \in \{0,1,2\}$

if $p_0 = p_1 = r$, then $\mu = \mu_r$ is a *symmetric* Bernoulli measure

μ_r is computable if and only if r is a computable real number.

Proposition (Cenzer, Porter)

Let μ_r be a symmetric Bernoulli measure on 3^ω for some $r \in [0, 1/2]$. Then the μ_r^{**}-measure of the collection of non-total continuous functions on 2^ω is 0 if $r \geq 1/4$ and is

$$\frac{1-4r}{(1-2r)^2}, \qquad \textit{if } r < 1/4.$$

When $r = 1/3$, this means that almost all functions are total.

Being in the Range

Theorem (Cenzer, Porter)

Let μ_r be a symmetric Bernoulli measure on 3^ω for some $r \in (0, 1/2]$ and let $y \in 2^\omega$. Then the μ_r^{**}-measure of the collection of continuous functions F such that $y \in \mathrm{ran}(F)$ is equal to

$$\frac{1-2r}{1-2r+r^2}.$$

When $r = 1/3$, this gives measure $3/4$.

Random Online Functions

For the symmetric Bernoulli measure $\mu_{1/2}$, random functions are represented by random sequences in 2^ω

Without delay, not every continuous function can be represented, for example, $F(x)(n) = x(2n)$.

Every random online function is total.

For any real y, the probability that $y \in \mathrm{ran}(F)$ is zero.

Results for Online Random Functions I

Theorem (Cenzer, Porter)

No computable real is in the range of an online random function.

Sketch: The probability p_n that $ran(F)$ hits $x{\upharpoonright}n$ is given by $p_{n+1} = p_n - \frac{1}{4}p_n^2$, which converges to zero.

Corollary

No online random function is onto.

Results for Online Random Functions II

Theorem (Cenzer, Porter)

Let F be an online random function and let $x \in 2^\omega$ code the representing function of F. If y is Martin-Löf random relative to x, then $F^{-1}(\{F(y)\})$ is a standard random closed set.

Sketch: The map Θ taking $x \oplus y$ to a representation of the closed set $F^{-1}(\{F(y)\})$ induces the uniform measure and preserves randomness.

Corollary

No online random function is one-to-one.

Results for Online Random Functions III

Theorem (Cenzer, Porter)

The range $\mathrm{ran}(F)$ *of an online random function* F *is not a standard random closed set.*

The proof uses martingales

Capacity

Definition

A *capacity* on $\mathcal{C}(2^\omega)$ is a function $\mathcal{T} : \mathcal{C}(2^\omega) \to [0,1]$ with $\mathcal{T}(\emptyset) = 0$ such that

1. $Q_1 \subseteq Q_2$ implies $\mathcal{T}(Q_1) \leq \mathcal{T}(Q_2)$.
2. For $n \geq 2$ and any $Q_1, \ldots, Q_n \in \mathcal{C}$

$$\mathcal{T}(\bigcap_{i=1}^{n} Q_i) \leq \sum \{(-1)^{|I|+1} \mathcal{T}(\bigcup_{i \in I} Q_i) : \emptyset \neq I \subseteq \{1, 2, \ldots, n\}\}.$$

3. If $Q = \bigcap_n Q_n$ and $Q_{n+1} \subseteq Q_n$ for all n, then $\mathcal{T}(Q) = \lim_{n\to\infty} \mathcal{T}(Q_n)$.

$\mathcal{T}$ is computable if it is computable on the family of clopen sets.

Choquet Capacity Theorem

Given a measure μ^* on the space $\mathcal{C}(2^\omega)$ of closed sets, define

$$\mathcal{T}_\mu(Q) = \mu^*(\{\mathcal{X} \in \mathcal{C}(2^\omega) : \mathcal{X} \cap \mathcal{S} \neq \emptyset\}).$$

$\mathcal{T}_\mu(Q)$ is the probability that a random closed set meets Q.

For $\mu_{1/3}$, $\mathcal{T}([\sigma]) = (2/3)^{|\sigma|}$.

Theorem (Effective Choquet Capacity Theorem, LMCS2011)

1. *For any computable probability measure μ on $\mathcal{C}(2^\omega)$, $\mathcal{T}_\mu$ is a computable capacity.*
2. *For any computable capacity $\mathcal{T}$ on $\mathcal{C}(2^\omega)$, there is a computable measure μ on the space of closed sets such that $\mathcal{T} = \mathcal{T}_\mu$.*

Previous Capacity Results

Let $\mathcal{T}_r = \mathcal{T}_{\mu_r}$.

Theorem (LCMS 2011)

1. *If* $r \geq 1 - \sqrt{2}/2$, *then* $\mathcal{T}_r(Q) = 0$ *for every* μ_r*-random* Q.
2. *If* $r < 1 - \sqrt{2}/2$, *then* $\mathcal{T}_r(Q) > 0$ *for every* μ_r*-random* Q.

Theorem (LCMS 2011)

1. *For any* Π^0_1 *class* Q, $\mathcal{T}_r(Q)$ *is upper semi-computable.*
2. *For any upper semi-computable real* q, *there is* Π^0_1 *class* Q *with capacity* q.

Theorem (LCMS 2011)

For any r, *there is an effectively closed set* Q *with positive capacity and with Lebesgue measure 0.*

From Functions to Capacities I

Theorem (Cenzer, Porter)

Let ν^{**} be a computable measure on $\mathcal{F}(2^\omega)$ such that every ν^{**}-random function is total. Then the function

$$\mathcal{T}(Q) = \nu^{**}(\{F \in \mathcal{F}(2^\omega) : \mathrm{ran}(F) \cap Q \neq \emptyset\})$$

is a computable capacity on $\mathcal{C}(2^\omega)$.

Sketch: The map Φ taking F to $\mathrm{ran}(F)$ is a Turing functional defined on a set of ν-measure one.

Then $\nu_\Phi = \nu(\Phi^{-1}(X))$ defines a computable measure and

$$\begin{aligned}\mathcal{T}(Q) &= \nu^{**}(\{F \in \mathcal{F}(2^\omega) : \mathrm{ran}(F) \cap Q \neq \emptyset\}) \\ &= \nu^*_\Phi(\{C \in \mathcal{C}(2^\omega) : \mathcal{C} \cap Q \neq \emptyset\}).\end{aligned}$$

From Functions to Capacities II

Proposition (Cenzer, Porter)

For $\nu = \mu_r$, the ν_Φ-random closed sets are not the standard random closed sets.

Sketch: $0^\infty \in ran(F)$ with positive probability under ν_Φ but cannot belong to a standard random closed set

Capacity and Complexity

Theorem (Cenzer, Porter)

Let μ^* be a computable measure on $\mathcal{C}(2^\omega)$ and $\mathcal{T}_\mu$ the computable capacity associated to μ. If x is a member of some μ^*-random closed set, then there is some c such that for all n

$$K(x{\upharpoonright}n) \geq -\log \mathcal{T}_\mu([X{\upharpoonright}n]) - c.$$

Sketch: Suppose that for every c, there exists n such that $K(x{\upharpoonright}n) < -\log \mathcal{T}_\mu([X{\upharpoonright}n]) - c$.

Then $\mathcal{U}_i = \{Q \in \mathcal{C}(2^\omega) : (\exists\sigma \in \widehat{S}_i)[Q \cap [\![\sigma]\!] \neq \emptyset]\}$ will be a μ^*-Martin-Löf test which Q fails if $x \in Q$.

The Range of an Online Random Function

Let $\mathcal{T}$ be the capacity defined above using the ranges of online random functions

Recall the computable sequence p_n with limit zero such that $\mathcal{T}([\sigma]) = p_n$ where $|\sigma| = n$.

Note that $f(n) = p_n$ is an *order function*.

Then we have:

Corollary

If x is in the range of a random online function, then for some c, $K(x{\upharpoonright}n) \geq -log\ p_n - c$ for all n. Thus x is complex.

Online Random Partial Functions

We introduce this notion in order to find a capacity which matches the capacity associated to the standard random closed sets.

Now $f(\sigma) = 2$ represents permanent divergence of the function along σ, rather than delay.

Theorem (Cenzer, Porter)

If μ_r is a computable symmetric Bernoulli measure on 3^ω, then the probability that a μ_r^{**}-random online partial function has non-empty domain is 0 if $r \geq 1/4$ and is

$$\frac{4r-1}{4r^2}, \qquad \text{if } r < 1/4.$$

Generalized Symmetric Bernoulli Measures

Given a computable sequence of rationals $\vec{r} = (r_i)_{i \in \omega}$ with $r_i \leq 1/2$ for every i, define a measure $\mu_{\vec{r}}$ on 3^ω such that for each n and each σ of length n,

- $\mu_{\vec{r}}(\sigma 0 \mid \sigma) = \mu(\sigma 1 \mid \sigma) = r_n \cdot \mu(\sigma)$ and
- $\mu_{\vec{r}}(\sigma 2 \mid \sigma) = (1 - 2r_n)\mu(\sigma)$.

The Main Theorem

Theorem (Cenzer, Porter)

Let $\mathcal{T}$ *be an computable capacity on* $\mathcal{C}(2^\omega)$ *and a computable sequence of rationals* $(p_i)_{i\in\omega}$ *such that*

(i) *for each* n, $\mathcal{T}([\sigma]) = p_n$ *for every* $\sigma \in 2^n$, *and*

(ii) $\lim_{n\to\infty} p_n = 0$.

Then there is a computable, generalized symmetric Bernoulli measure $\mu_{\vec{r}}$ *on* 3^ω *such that the ranges of the* $\mu_{\vec{r}}^{**}$*-random online partial functions are precisely the random closed sets associated with the capacity* $\mathcal{T}$. *Moreover, in the case that* $\lim_{n\to\infty} \frac{p_{n+1}}{p_n} = p$ *for some* $p \in [0,1]$, *we have* $\lim_{n\to\infty} r_n = \frac{p}{2}$.

Standard Random Closed Sets

Starting with the standard capacity where $p_n = (\frac{2}{3})^n$, we obtain

Theorem (Cenzer, Porter)

Let $\vec{r} = (r_i)_{i\in\omega}$ be defined by

$$r_i = \frac{2/3}{1 + \sqrt{1 - \left(\frac{2}{3}\right)^i}}.$$

Then the family of ranges of the $\mu^{**}_{\vec{r}}$-random online partial functions is equal to the family of the standard random closed sets.

Sketch: The proof of the previous result gives us

$$r_{n+1} = \frac{p_{n+1}}{p_n(1 + \sqrt{1 - p_{n+1}})}.$$

Future Topics

- Random online functions with finite delay
- Average values (integrals) of random functions

The End

Thank You