The role of randomness in reverse mathematics

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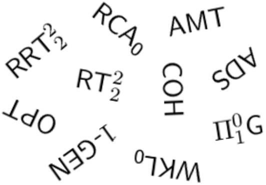

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THEOREMS AS PROBLEMS

Look at "ordinary" theorems:

- (König's lemma)
  Every infinite, finitely branching tree has an infinite path.
- (Ramsey's theorem)
  Every k-coloring has an infinite monochromatic subset.
- (The atomic model theorem)
  Every complete atomic theory has an atomic model.
- ...

THEOREMS AS PROBLEMS

Many theorems P are of the form

$$(\forall X)[\Phi(X) \to (\exists Y)\Psi(X, Y)]$$

where Φ and Ψ are arithmetic formulas.

We may think of P as a class of problems.

- An X such that $\Phi(X)$ holds is an instance.
- A Y such that $\Psi(X, Y)$ holds is a solution to X.

STRENGTH OF A THEOREM

Some theorems are more effective than others.

Theorem (Intermediate value theorem)
For every continuous function f over $[a,b]$ and every $y \in [f(a), f(b)]$, there is some $x \in [a,b]$ such that $f(x) = y$.

Theorem (König's lemma)
Every infinite, finitely branching tree has an infinite path.

STRENGTH OF A THEOREM

Provability strength

- Reverse mathematics
- Intuitionistic reverse mathematics

Computational strength

- Computable reducibility
- Uniform reducibility

Provability approach

REVERSE MATHEMATICS

Goal

Determine which axioms are required to prove ordinary theorems in reverse mathematics.

- Simpler proofs
- More insights

Subsystems of second-order arithmetic.

BASE THEORY RCA_0

- Basic Peano axioms
- Σ^0_1 induction scheme

$$(\varphi(0) \wedge \forall n.(\varphi(n) \to \varphi(n+1))) \to \forall n.\varphi(n)$$

  where $\varphi(n)$ is any Σ^0_1 formula of L_2

- Δ^0_1 comprehension scheme

$$\forall n(\varphi(n) \leftrightarrow \psi(n)) \to \exists X.\forall n.(x \in X \leftrightarrow \varphi(n))$$

  where $\varphi(n)$ is any Σ^0_1 formula of L_2 in which X does not occur freely and $\psi(n)$ is any Π^0_1 formula of L_2.

HOW TO THINK ABOUT RCA_0 ?

RCA_0 captures computable mathematics

RCA_0 has model $\mathcal{M} = \{\omega, S, <, +, \cdot\}$ where

- ω is the set of the standard integers
- $S = \{X \in 2^\omega : X \text{ is computable }\}$ is the second-order part

NON-PROVABILITY OVER RCA_0

Let P be a statement.

How to prove that $\mathsf{RCA}_0 \nvdash \mathsf{P}$?

A method: Exhibit a computable instance I of P which admits no computable solution.

NON-PROVABILITY OVER RCA_0

Let $\mathcal{M}$ be the model of RCA_0 whose second-order part are the computable sets.

- $\mathcal{M} \models \mathsf{RCA}_0$;
- Because I is computable, $I \in \mathcal{M}$;
- Because I does not have a computable solution, $\mathcal{M} \not\models \mathsf{P}$.

Therefore $\mathsf{RCA}_0 \nvdash \mathsf{P}$.

PROBABILISTIC SOLUTIONS

Are there probabilistic algorithms to solve instances with no computable solution?

PROBABILISTIC SOLUTIONS

Definition (n-RAN)

"For every set X, there is a Martin-Löf random real relative to $X^{(n-1)}$".

Given a statement P, does $\mathsf{RCA}_0 \vdash n\text{-}\mathsf{RAN} \to \mathsf{P}$ for some n ?

PROBABILISTIC SOLUTIONS

Usually not

Definition (No randomized algorithm)

A statement P has the NRA property if it has a computable instance I such that

$$\mu\{X : X \text{ computes a solution to } I\} = 0$$

PROBABILISTIC SOLUTIONS

If P has the NRA property
then $\mathsf{RCA}_0 \nvdash n\text{-}\mathsf{RAN} \to \mathsf{P}$ for every n.

If P has the NRA property and $\mathsf{RCA}_0 \vdash \mathsf{Q} \to \mathsf{P}$
then Q has the NRA property.

PROBABILISTIC SOLUTIONS

Many weak statements not provable over RCA_0 have the NRA property.

Intuition

- Many proofs of a computable P-instance with no computable solutions are diagonalizations.
- Many diagonalizations can be done by block, defeating positive measure of oracles.

DIAGONAL NON-COMPUTABILITY

Definition (Diagonal non-computability)

A function f is DNC relative to X if $(\forall e)[f(e) \neq \Phi_e^X(e)]$

- Simplest example of non-computable function.
- Cantor's diagonal argument.
- Unifying framework for comparing theorems.

DIAGONAL NON-COMPUTABILITY

Theorem

The following are computably equivalent:

- *DNC functions relative to X*
- *Infinite subset of X-Martin-Löf randoms*
- *Escaping X-c.e. sets of computably bounded size*

n-DNC

For every set X, there is a function DNC relative to $X^{(n-1)}$.

DIAGONAL NON-COMPUTABILITY

Theorem

The following are computably equivalent:

- $\{0,1\}$*-valued DNC functions relative to* X
- *Computing an infinite path through an* X*-computable infinite binary tree*
- *Choosing between two* $\Pi_2^{0,X}$ *statements*

n-DNC$_2$

For every set X, there is a $\{0,1\}$-valued function DNC relative to $X^{(n-1)}$.

DIAGONAL NON-COMPUTABILITY

Theorem
$\mathsf{RCA}_0 \vdash n\text{-}\mathsf{RAN} \rightarrow \mathsf{n\text{-}DNC}$

Hint: To define $f(n)$, pick a number at random in $[0, 2^{n+2}]$.

Theorem (Jockusch & Soare)
n-DNC_2 *has the NRA property.*

Hint: A finite range enables us to apply the pigeonhole principle and defeat a block of oracles.

Ramsey's theorem

RAMSEY'S THEORY

Given some size s, every sufficiently large collection of objects has a sub-collection of size s, whose objects satisfy some structural properties.

RAMSEY'S THEOREM

Definition

Given a coloring $f : [\mathbb{N}]^n \to k$, a set H is f-homogeneous if there exists a color $i < k$ such that $f([H]^n) = i$.

RT^n_k (Ramsey's theorem)

Every coloring $f : [\mathbb{N}]^n \to k$ has an infinite f-homogeneous set.

COHESIVENESS

Definition

Given a sequence of sets $R_0, R_1, \ldots$, an infinite set C is $\vec{R}$-cohesive if for every i, $C \subseteq^* R_i$ or $C \subseteq^* \overline{R_i}$.

COH (Cohesiveness)

Every sequence of sets $R_0, R_1, \ldots$ has an $\vec{R}$-cohesive set.

COHESIVENESS

Theorem (Jockusch & Stephan)

The following are computably equivalent

- COH
- *For every set X, there is a $\{0,1\}$-valued function DNC relative to X'.*

Corollary (Jockusch & Soare)

COH *has the NRA property.*

THE ATOMIC MODEL THEOREM

AMT (Atomic model theorem)

Every complete atomic theory has an atomic model.

Theorem (Hirschfeldt, Shore, Slaman & Conidis)

The following are computably equivalent:

- AMT
- *For every Δ^0_2 function f, there exists a function g such that $f(x) \leq g(x)$ for infinitely many x.*

THE ATOMIC MODEL THEOREM

Theorem (Kurtz)

AMT *has the NRA property.*

Hint: $\emptyset'$ is uniformly almost everywhere dominating.

THE RAINBOW RAMSEY THEOREM

Definition (k-bounded function)

A coloring function $\mathbb{N}^n \to \mathbb{N}$ is k-bounded if $|\{x \in \mathbb{N}^n : f(x) = c\}| \leq k$ for every color c.

RRT^n_k (Rainbow Ramsey theorem)

For every k-bounded coloring function $f : \mathbb{N}^n \to \mathbb{N}$ there is an infinite set H such that $f \upharpoonright H^n$ is injective.

THE RAINBOW RAMSEY THEOREM

Theorem (Csima & Mileti)

$\mathsf{RCA}_0 \vdash \mathsf{2\text{-}RAN} \to \mathsf{RRT}^2_2$

Theorem (Miller)

$\mathsf{RCA}_0 \vdash \mathsf{RRT}^2_2 \leftrightarrow \mathsf{2\text{-}DNC}$

Hint: The set of "bad" one-point extensions is a computably bounded $\emptyset'$-c.e. set.

THE RAINBOW RAMSEY THEOREM

Theorem (Bienvenu, Patey & Shafer)

RRT^3_2 *has the NRA property.*

Hint: RRT^3_2 implies the atomic model theorem over RCA_0.

THE FINITE INTERSECTION PROPERTY

Definition

A sequence of set $A_0, A_1, \ldots$ has the FIP if the intersection of finitely many sets is non-empty.

FIP (Finite intersection property)

Every sequence of sets has a maximal subsequence having the FIP.

- Equivalent to the axiom of choice in set theory.

THE FINITE INTERSECTION PROPERTY

Definition

Fix a set of strings S. A real G meets S if it has some initial segment in S. A real G avoids S is it has an initial segment with no extension in S. A real X is n-generic if it meets or avoids every Σ^0_n set of strings.

n-GEN (n-genericity)

For every set X, there is a real n-generic relative to X.

Theorem (Cholak, Downey, Diamondstone, Greenberg, Igusa & Turetsky)

$\mathsf{RCA}_0 \vdash \mathsf{FIP} \leftrightarrow 1\text{-}\mathsf{GEN}$

THE FINITE INTERSECTION PROPERTY

Theorem (Kurtz, Kautz)

$\mathsf{RCA}_0 \vdash \text{2-}\mathsf{RAN} \to \mathsf{FIP}$

Hint: Use a fireworks argument.

Is 2-RAN needed? What about 2-DNC?

Conclusion

- Few theorems studied in reverse mathematics and not provable over RCA_0 admit probabilistic algorithms.
- All known examples have natural computability-theoretic characterization and admit a universal instance.
- Is 1-genericity a reverse mathematical consequence of the rainbow Ramsey theorem for pairs?

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Questions

Thank you for listening!