Algorithmic Randomness and Constructive/Computable Mathematics

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Varieties of Algorithmic Information
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Slides available at
www.personal.psu.edu/jmr71/

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Motivation

Randomness and constructive mathematics

Algorithmic randomness

A formulation of randomness through computation.

Constructive mathematics

A foundation of mathematics preserving computational meaning.

Observation

These two subjects must be connected. **...but how are they connected?**

Constructive measure theory and probability

Bishop 1967

Any constructive approach to mathematics will find a crucial test in the ability to assimilate the intricate body of mathematical thought called measure theory. [...] It was recognized by Lebesgue, Borel, and other pioneers in abstract function theory that the mathematics they were creating relied, in a way almost unique at the time, on set-theoretic methods, leading to results whose constructive content was problematical.

- Brouwer 1919, et al.
- Šanin 1962, et al.
- Demuth 1965, et al.
- Bishop 1967, et al.
- Martin-Löf 1970
- etc.

Also...

- Friedman / Ko 1982 (Poly-time analysis)
- Pour-El / Richards 1989 (Comp. analysis)
- Yu / Simpson 1990, et al. (Reverse math)
- Edalat 1995, et al. (Domain theory)
- Weihrauch 1997, et al. (Type-2 effectivity)
- etc.[1]

[1] Apologies to any tradition I left out.

Algorithmic randomness and "constructive null sets"

Martin-Löf 1966 (Emphasis mine)

In this paper it is shown that the random elements as defined by Kolmogorov possess all conceivable statistical properties of randomness. They can equivalently be considered as the elements which withstand a certain universal stochasticity test. The definition is extended to infinite binary sequences and it is shown that the non random sequences form a ***maximal constructive null set****.*

Schnorr 1969 (Emphasis mine)

Martin-Löf has defined random sequences to be those sequences which withstand a certain universal stochasticity test. On the other hand one can define a sequence to be random if it is not contained in any ***[set] of measure zero in the sense of Brouwer****. Both definitions imply that these random sequences possess all statistical properties which can be checked by algorithms. We draw a comparison between the two concepts of* ***constructive null sets*** *and prove that they induce concepts of randomness which are not equivalent. The union of all* ***[sets] of measure zero in the sense of Brouwer*** *is a proper subset of the* ***universal constructive null set*** *defined by Martin-Löf.*

Randomness characterized by a.e. theorems

Randomness characterization theorem template

x is ________ random iff x satisfies ______________ for all computable ________.

MLR and Lebesgue's thm (Demuth; Bratkka / Miller / Nies)

x is Martin-Löf random iff
$f'(x)$ exists for all computable $f\colon [0,1] \to \mathbb{R}$ of bounded variation.

SR and Lebesgue differentiation thm (Pathak / Rojas / Simpson; Rute)

x is Schnorr random iff
$\frac{1}{2r}\int_{x-r}^{x+r} f(y)\,dy$ converges as $r \to 0$ for all L^1-computable functions f.

Questions

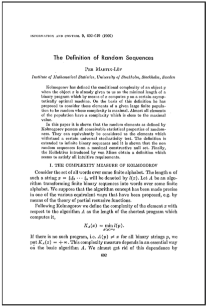
- What do such results say about constructive measure theory?
- What does constructive measure theory say about such results?

Algorithmic randomness

Effectively open sets

- Recall: An **effectively open (Σ^0_1) set** is a computable union of basic open sets (i.e., cylinder sets for $2^{\mathbb{N}}$ or rational intervals for $[0,1]$).

Martin-Löf randomness



INFORMATION AND CONTROL 9, 602-619 (1966)

The Definition of Random Sequences

Per Martin-Löf

Institute of Mathematical Statistics, University of Stockholm, Stockholm, Sweden

Kolmogorov has defined the conditional complexity of an object y when the object x is already given to us as the minimal length of a binary program which by means of x computes y on a certain asymptotically optimal machine. On the basis of this definition he has proposed to consider those elements of a given large finite population to be random whose complexity is maximal. Almost all elements of the population have a complexity which is close to the maximal value.

In this paper it is shown that the random elements as defined by Kolmogorov possess all conceivable statistical properties of randomness. They can equivalently be considered as the elements which withstand a certain universal stochasticity test. The definition is extended to infinite binary sequences and it is shown that the non random sequences form a maximal constructive null set. Finally, the Kollektivs introduced by von Mises obtain a definition which seems to satisfy all intuitive requirements.

I. THE COMPLEXITY MEASURE OF KOLMOGOROV

Consider the set of all words over some finite alphabet. The length n of such a string $x = \xi_1\xi_2 \cdots \xi_n$ will be denoted by $l(x)$. Let A be an algorithm transforming finite binary sequences into words over some finite alphabet. We suppose that the algorithm concept has been made precise in one of the various equivalent ways that have been proposed, e.g. by means of the theory of partial recursive functions.

Following Kolmogorov we define the complexity of the element x with respect to the algorithm A as the length of the shortest program which computes it,

$$K_A(x) = \min_{A(p)=x} l(p).$$

If there is no such program, i.e. $A(p) \neq x$ for all binary strings p, we put $K_A(x) = +\infty$. This complexity measure depends in an essential way on the basic algorithm A. We almost get rid of this dependence by

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Definition (Martin-Löf [1966])

- A **Martin-Löf test** is a computable sequence of effectively open sets U_n such that $\mu(U_n) \leqslant 2^{-n}$.
- A **Martin-Löf null set** is a subset of $\bigcap_n U_n$ for some Martin-Löf test U_n.
- A **Martin-Löf random** is a point x not in any Martin-Löf null set.

- Clearly no computable real is Martin-Löf random.

Theorem (Martin-Löf [1966])

There is a universal Martin-Löf null set covering all other Martin-Löf null sets.

Schnorr randomness

Definition (Schnorr [1969])

- A **Schnorr test** is a computable sequence of effectively open sets U_n such that $\mu(U_n) \leqslant 2^{-n}$ **and $\mu(U_n)$ is computable in n**.
- A **Schnorr null set** is a subset of $\bigcap_n U_n$ for some Schnorr test U_n.
- A **Schnorr random** is a point x not in any Schnorr null set.

- Clearly no computable real is Schnorr random.
- Clearly every Schnorr null set is a Martin-Löf null set.
- Hence, every Martin-Löf random is Schnorr random.

Schnorr randomness

Theorem (Schnorr [1969])

1. Schnorr randomness is strictly weaker than Martin-Löf randomness.
2. For every Schnorr null set N, there is a computable point not in N.
3. Hence, there is no universal Schnorr null set.

- The second item is an effective version of the following.

Theorem

Every set of measure one contains a point.

Constructive measure theory

A mathematical dialog

The characters

NAÏVE	An enthusiastic new student of constructivism.
CONSTRUCT	A (Bishop-style) constructivist.
INT	A Brouwerian intuitionist.
RUSS	A constructivist in the Russian school.
COMPUTE	A computability theorist / computable analyst.
RANDOM	An algorithmic randomist.

Constructive mathematics

NAÏVE There are many approaches to constructive mathematics.

CONSTRUCT Yes. However, they share a common principle:

The constructive principle

A constructive proof of "there exists a object x such that $P(x)$", **provides an algorithm constructing such an x** (along with a proof that P holds of x).

INT, RUSS Yes, that sounds about right.

COMPUTE Is constructive mathematics then consistent with the following?

Church's thesis (in constructive math)

All functions (and therefore all real numbers) are computable.

RUSS I accept this. CONSTRUCT It is not inconsistent with my beliefs.

INT I don't accept this. (But I also am not saying there exists a noncomputable function either.)

Computable interpretation of constructive math

COMPUTE It seems that constructive math has a computable interpretation.

CONSTRUCT When I say a real x exists...

COMPUTE ...that is to say a computable real x exists.

CONSTRUCT An open set U exists...

COMPUTE An effectively open (Σ^0_1) set U exists...

CONSTRUCT The sequence x_n of reals (constructively) converges.

COMPUTE The sequence x_n of reals has a limit computable uniformly from x_n.

Uncountability of the continuum

CONSTRUCT One can constructively prove that the set of reals is uncountable.

NAÏVE Wait, one can prove that? Doesn't that violate Church's thesis?

CONSTRUCT No it does not. Cantor's argument is constructive.

COMPUTE There is no computable enumeration of the computable reals.

NAÏVE What about the classical result that the continuum is not null?

CONSTRUCT That is a good question.

COMPUTE That is like asking if the computable reals do not form an effective null set.

RANDOM But which type of effective null set?

COMPUTE Yes, how does a constructivist define a null set?

Cover definition of null sets

NAÏVE I have an idea. Let's define it using covers (or coverings) just like in classical Lebesgue measure theory.

NAÏVE's definition of a constructive null set

A set $N \subseteq [0,1]$ is **(constructively) null** if for all $\varepsilon > 0$ there is an ε-cover of N.

- An **ε-cover** of N is a sequence of rational intervals $\{(a_k, b_k)\}_{k\in\mathbb{N}}$ such that

$$N \subseteq \bigcup_{k\in\mathbb{N}} (a_k, b_k) \quad \text{and} \quad \sum_{k\in\mathbb{N}} |b_k - a_k| \leqslant \varepsilon.$$

- Note $\sum_{k\in\mathbb{N}} |b_k - a_k|$ is not assumed to constructively exist.
- Instead, $\sum_{k\in\mathbb{N}} |b_k - a_k| \leqslant \varepsilon$ is short-hand for $\forall n \; \sum_{k=0}^{n-1} |b_k - a_k| \leqslant \varepsilon$.

CONSTRUCT You are not the first to suggest this approach.

COMPUTE That is to say that a null set is a...

RANDOM **...a Martin-Löf null set!**

First paradox of singular covers

RUSS Zaslavskiĭ and Ceĭtin [1962] showed there exist **singular covers**, that is ε-covers of the computable reals for each $\varepsilon < 1$.

COMPUTE Similarly, Kreisel and Lacombe [1957] showed that the computable reals are inside a Σ^0_1 set of arbitrarily small measure.

RANDOM Also, Martin-Löf [1966] showed the computable reals are a Martin-Löf null set.

NAÏVE Is this a problem? CONSTRUCT Consider this argument:

First paradox of singular covers

1. Assume $[0,1]$ is not a NAÏVE null set.
2. The set of computable reals is a NAÏVE null set (via singular covers).
3. Hence, $[0,1]$ is not made up of only computable reals.
4. Therefore, "$[0,1]$ is not a NAÏVE null set" contradicts Church's thesis.

CONSTRUCT Martin-Löf and Beeson have suggested this is a reason to deny Church's thesis.

Second paradox of singular covers

NAÏVE What about the result that every set of measure one has a point?

CONSTRUCT It gets even worse.

Second paradox of singular covers

1. Assume, every set of measure one contains a point.
2. The unit interval $[0,1]$ has measure one. (Geometry, definition of measure.)
3. There is a NAÏVE null set N containing all computable reals.
4. Hence, $[0,1] \setminus N$ is a set of measure one. (Basic measure theory.)
5. Therefore, there exists a non-computable point.

COMPUTE A constructive proof of 1 would let us construct a noncomputable point.

Regular covers

CONSTRUCT I have another solution. Let us use a more constructive null set.

CONSTRUCT's definition of a constructive null set

A set N is **(constructively) null** if for all $\varepsilon > 0$ there is a regular ε-cover of N.

- A **regular ε-cover** of N is a sequence of rational intervals $\{(a_k, b_k)\}_{k\in\mathbb{N}}$ such that

$$N \subseteq \bigcup_{k\in\mathbb{N}} (a_k, b_k), \quad \sum_{k\in\mathbb{N}} |b_k - a_k| \text{ converges,} \quad \text{and} \quad \sum_{k\in\mathbb{N}} |b_k - a_k| \leqslant \varepsilon.$$

RANDOM That is like a Schnorr null set!

CONSTRUCT Now it is constructively true that every full set contains a point.

RANDOM Yes, one can compute a point not in a Schnorr null set.

CONSTRUCT It is not surprising that Brouwer, Bishop, Demuth, and others early constructivists used this definition of null set.

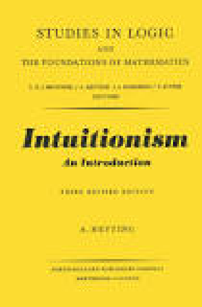
Epilogue: The point-free approach

Point free I overheard you all talking. Let me suggest another "modern" approach. Instead of thinking about measurable sets as actual sets containing points, just think about them as formal objects in some Boolean algebra with a metric space structure. Then a null set is just the bottom element of this Boolean algebra. Moreover, all the basic facts of measure theory, for example the strong law of large numbers or even the pointwise ergodic theorem can be expressed in this form. Some (but not all) of these theorems are even constructively provable.

Naïve I wonder how Point free's approach relates to that of myself and that of Construct?

Survey of constructive measure theory

L.E.J. Brouwer (Intuitionist school)



- Brouwer [1919] wrote a paper on constructive measure theory.

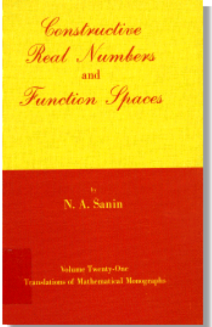
Brouwer's definition of measurable set and null set

- An open set U is **measurable** if $\mu(U)$ exists.
- A set N is **null** if it is included in a measurable open set of arbitrarily small measure.
- A set Q to **measurable** if for every n there is a measurable open set U_n and a basic set V_n such that

$$Q \triangle V_n \subseteq U_n \quad \text{and} \quad \mu(U_n) \leqslant 2^{-n}.$$

- Notice Brouwer's null sets are like Schnorr's null sets.
- Also U_n is like a Schnorr test.
- See Heyting's book (left) for a presentation of intuitionistic measure theory.

N.A. Šanin (Russian school)



- Šanin's 1962 book developed a theory of constructive measurable sets and measurable/integrable functions.

Šanin's definition of measurable set

- Consider the metric $\rho(A,B) = \mu(A \triangle B)$ on basic sets.
- This generates a **constructive metric space**.
- A **measurable set** is a (constructive) point in this metric space.

- This approach avoids discussions of null sets.
- Šanin's student Kosovskiĭ developed constructive probability theory using this approach.

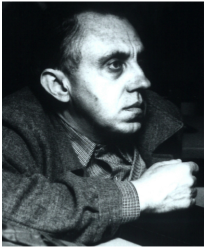
Zaslavskiĭ and Ceĭtin (Russian school)

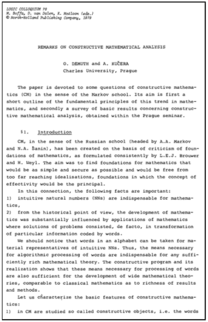
- While Zaslavskiĭ and Ceĭtin [1962] focused on the pathological consequences of singular coverings.
- However, they include this insightful remark.

Zaslavskiĭ and Ceĭtin [1962] (Emphasis in original)

*We call a covering Φ **regular** if the sequence of numbers $\sum_{k=0}^{n} |\Phi_k|$ is constructively convergent as $n \to \infty$. The set $\mathcal{M}$ of [constructive real numbers] will be said to be a **set of measure zero** if for arbitrary ε there can be realized a regular ε-bounded covering by intervals of the set. [...] Consequently, in spite of the existence of constructive singular coverings, it is possible to give a reasonable definition of the constructive concept of a set of measure zero. Other concepts of the constructive theory of measure can be defined in a similar way.*

O. Demuth (Russian school)





LOGIC COLLOQUIUM 78
M. Boffa, D. van Dalen, K. McAloon (eds.)
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REMARKS ON CONSTRUCTIVE MATHEMATICAL ANALYSIS

O. DEMUTH and A. KUČERA
Charles University, Prague

The paper is devoted to some questions of constructive mathematics (CM) in the sense of the Markov school. Its aim is first a short outline of the fundamental principles of this trend in mathematics, and secondly a survey of basic results concerning constructive mathematical analysis, obtained within the Prague seminar.

§1. Introduction

CM, in the sense of the Russian school (headed by A.A. Markov and N.A. Šanin), has been created on the basis of criticism of foundations of mathematics, as formulated consistently by L.E.J. Brouwer and H. Weyl. The aim was to find foundations for mathematics that would be as simple and secure as possible and would be free from too far reaching idealisations, foundations in which the concept of effectivity would be the principal.

In this connection, the following facts are important:
1) intuitive natural numbers (NNs) are indispensable for mathematics,
2) from the historical point of view, the development of mathematics was substantially influenced by applications of mathematics where solutions of problems consisted, de facto, in transformation of particular information coded by words.

We should notice that words in an alphabet can be taken for material representatives of intuitive NNs. Thus, the means necessary for algorithmic processing of words are indispensable for any sufficiently rich mathematical theory. The constructive program and its realization shows that these means necessary for processing of words are also sufficient for the development of wide mathematical theories, comparable to classical mathematics as to richness of results and methods.

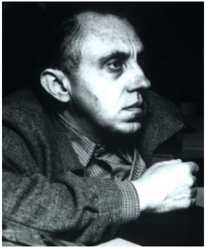
Let us characterize the basic features of constructive mathematics:
1) in CM are studied so called constructive objects, i.e. the words

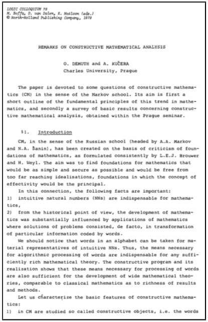
- Since 1965, Demuth wrote prolifically on constructive measure theory (see Demuth / Kučera [1979]).
- While Demuth's definitions of null set and measurable set agree with those of Brouwer [1919] and Zaslavskiĭ / Ceĭtin [1962], his definitions also borrow ideas from Šanin [1962].

Demuth's definition null set and measurable set

- **Null sets** are defined via regular ε-coverings.
- A function $f\colon [0,1] \to \mathbb{R}$ is **integrable** if $f = \lim_n f_n$ outside of a null set for a sequence of simple functions f_n such that $\forall n \geqslant m \quad \|f_m - f_n\|_{L^1} \leqslant 2^{-n}$.
- A set Q is **measurable** if $\mathbf{1}_Q$ is integrable.

O. Demuth (Russian school)





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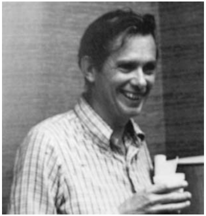
1) intuitive natural numbers (NNs) are indispensable for mathematics,
2) from the historical point of view, the development of mathematics was substantially influenced by applications of mathematics where solutions of problems consisted, de facto, in transformation of particular information coded by words.

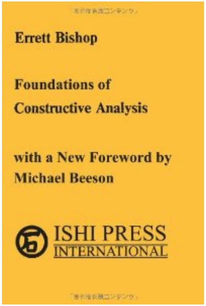
We should notice that words in an alphabet can be taken for material representatives of intuitive NNs. Thus, the means necessary for algorithmic processing of words are indispensable for any sufficiently rich mathematical theory. The constructive program and its realization shows that these means necessary for processing of words are also sufficient for the development of wide mathematical theories, comparable to classical mathematics as to richness of results and methods.

Let us characterize the basic features of constructive mathematics:
1) in CM are studied so called constructive objects, i.e. the words

- Like the other Russian constructivists, Demuth took the continuum to be the computable reals, hence his measurable sets where actually sets of computable reals.
- Demuth also worked with Martin-Löf null sets (independently of Martin-Löf).
- Martin-Löf null sets were important in his investigation of the differentiability of functions of bounded variation (which is not a constructive theorem).
- In this case, to avoid some of the paradoxes of singular covers, Demuth enlarged the space to incluse all $\emptyset'$-computable reals.
- See Kučera/Slaman [2001, Rmk. 3.5] or Porter/Kučera/Nies [201x] for more on Demuth's work in randomness.

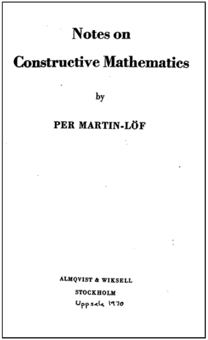
Bishop's School





- Bishop's 1967 book (and his later works) had a strong influence on constructive mathematics.
- He and his students wrote a lot on measure theory.
- His definitions of null set and measurable set agree with those of Brouwer and Demuth (although the definitions themselves are very different).

Martin-Löf


Notes on

Constructive Mathematics

by

PER MARTIN-LÖF

ALMQVIST & WIKSELL
STOCKHOLM
Uppsala 1970

- Martin-Löf is known for his early work in randomness and his later work on constructive type theory.
- In 1970 (the transitionary period) he wrote a book on constructive mathematics, which includes a chapter on measure theory.

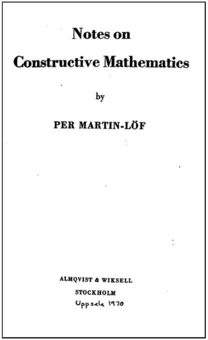
Martin-Löf's definition of measurable set

A set Q to **measurable** if for every n there is an open set U_n and a basic set V_n such that

$$Q \triangle V_n \subseteq U_n \quad \text{and} \quad \mu(U_n) \leqslant 2^{-n}.$$

- Unlike Brouwer's definition, $\mu(U_n)$ need not exist.
- Hence, U_n is a Martin-Löf test.
- Martin-Löf knew what this entailed...

Martin-Löf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PER MARTIN-LÖF

ALMQVIST & WIKSELL
STOCKHOLM
Uppsala 1970

Martin-Löf [1970]

There are several reasons why we have chosen a more inclusive definition of measurability than Brouwer did. First of all, the problem has always been to find a consistent extension of the measure, first defined for simple sets only, which goes as far as possible. Our extension, although going further than Brouwer's entails no departure from the constructive standpoint.

Secondly, the fact that our definition allows the construction of an inner limit set of measure zero which contains all constructive points, although troublesome to those whose continuum consists of constructive points only, is in full agreement with the intuitionistic concept of the continuum as a medium of free choice.

Thirdly, the definition we have adopted enables us to prove a new theorem [existence of a universal constructive null set] which may serve as a justification of the notion of a random sequence conceived by von Mises and elaborated by Wald and Church 1940.

Yu and Simpson (Reverse Math)

- RCA_0, while not constructive, has a computable interpretation.
- Reverse math determines the minimum axioms needed to prove a theorem (over RCA_0).
- WWKL (Weak-Weak König's Lemma) says that every closed set of positive measure contains a point.[2]

Theorem (Yu/Simpson)

In RCA_0, the following are equivalent:

- WWKL
- Every sequence of intervals (a_n, b_n) covering $[0,1]$ satisfies $\sum_{n=0}^{\infty}(b_n - a_n) \geqslant 1$.
- If $U, V \subseteq \{0,1\}^{\mathbb{N}}$ are disjoint open sets such that $U \cup V = \{0,1\}^{\mathbb{N}}$ then $\mu(U) + \mu(V) = 1$.

[2] Usually WWKL is defined in terms of "trees of positive measure."

Yu and Simpson (Reverse Math)

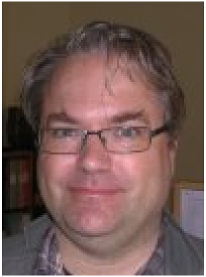
- WWKL is also equivalent to other theorems:
  - For all X, there is a Martin-Löf random relative to X.
  - A variant of the monotone convergence theorem.
  - A variant of the Vitali covering theorem.

Reverse math definition of a null set

- Null sets are defined via (not necessarily regular) ε-covers.[3]
- Measurable sets and functions are defined in a variety of ways depending on the paper.

[3] The details can very slightly from paper to paper, but I believe this is the intended definition.

Coquand and Palmgren (Point-free constructivism)



- "Point-free" approaches to constructive measure theory are similar to those of Šanin and Kosovskiĭ.

Coquand and Palmgren [2009][4]

- Start with a Boolean ring R and a measure μ satisfying
  - $(\forall x \in R) \quad \mu(x \vee y) = \mu(x) + \mu(y) - \mu(x \wedge y)$
  - $(\forall x \in R) \quad \mu(x) > 0 \quad \leftrightarrow \quad x \neq 0.$
- E.g. the Boolean ring of clopen sets of $2^{\mathbb{N}}$ with the Lebesgue measure.
- Extend to a complete metric space via the metric $\rho(x,y) = \mu(x+y)$.

- Using this approach they state and constructively prove Kolmogorov's 0-1 law, the first Borel-Cantelli lemma, and the strong law of large numbers.

[4] This is the first of two approaches to measurable sets in their paper.

Three approaches to measurable objects

and why they are basically the same

The objects of measure theory

- There are five main types of objects in Lebesgue measure theory:
  - Null sets
  - Measurable sets
  - Measurable functions
  - Integrable functions
  - Almost-uniform convergence
    - is classically equivalent to a.e. convergence on a prob. space.

- There are three main approaches to defining such objects:
  - Brouwer/Schnorr approach:
    - Define modulo a Schnorr null set
  - Martin-Löf approach:
    - Define modulo a Martin-Löf null set
  - Point-free approach:
    - Define modulo a null set using a comp. metric space

Brouwer/Schnorr approach

- A **basic set** is a finite union of basic open sets. (A clopen set on $2^{\mathbb{N}}$.)

Brouwer/Schnorr measurable set (Brouwer [1919], Heyting [1956])

A set Q to **Brouwer/Schnorr effectively measurable** if there is a computable sequence of basic sets (B_n) such that

$$Q \triangle B_n \subseteq U_n \quad \text{for some Schnorr test } (U_n).$$

Same or equivalent approaches can be found in:

- Brouwer et al. (Intuitionism)
- Demuth et al. (Russian school)
- Bishop et al. (Bishop-style)
- Pathak/Rojas/Simpson, Miyabe, Rute (Randomness and Schnorr layerwise computability)

Martin-Löf approach

Martin-Löf measurable set (Martin-Löf's book [1970])

A set Q is **Martin-Löf effectively measurable** if there is a computable sequence of basic sets (B_n) such that

$$Q \triangle B_n \subseteq U_n \quad \text{for some Martin-Löf test } (U_n).$$

Same or equivalent approaches can be found in:

- Martin-Löf
- Yu/Simpson et al. (Reverse math)
- Edalat (Domain theory)
- Hoyrup/Rojas (Randomness and layerwise computability)
- Pathak (Randomness)

Point-free approach

- Let ρ be the metric first defined on the basic sets via $\rho(A, B) = \mu(A \triangle B)$.
- The completion of ρ is the Boolean algebra of measurable sets mod null.

Point-free measurable set (Šanin [1962])

A set Q is **point-free effectively measurable** if there is a computable sequence of basic sets (B_n) such that

$$\mu(Q \triangle B_n) \leqslant 2^{-n}.$$

Same or equivalent approaches can be found in:

- Šanin, Kosovskii (Russian school)
- Friedman/Ko (polytime analysis)
- Pour-El/Richards, Wu/Ding, Edalat, etc. (Computable analysis)
- Yu/Simpson (Reverse math)
- Coquand/Palmagren, Spitters (modern constructive math)
- Any paper on L^1-computability

Point-free approach

- In the point-free approach, such questions don't make sense:
  - Is $1/3$ a point in the set Q?
  - Does Q contain a point?
- This is because Q is not an actual set, just a formal object.

- With a little work, all the usual a.e. pointwise convergence theorems can be expressed in a point-free form.
- (Technically, these formulations use almost-uniform convergence in place of a.e. convergence, but that is how the results are constructively proved anyway.)

Moral equivalence of the three approaches

Theorem (Constructive)

1. Every Brouwer/Schnorr measurable set is Martin-Löf measurable.
2. Every Martin-Löf measurable set is point-free measurable.
3. Every point-free measurable set is a.e. equal to some Brouwer/Schnorr measurable set.

- The same holds for integrable/measurable functions and almost-uniform convergence.

- Therefore, for many theorems, including the a.e. pointwise convergence theorems, it doesn't matter which definitions we use.
- In particular, it doesn't hurt to use the Brouwer/Schnorr versions.

Connecting randomness and analysis

Our questions

Questions

1. What does a constructive/computable result say about randomness?
2. What does a randomness result say about constructivity/computability?

A coincidence? I think not.

Constructive theorems

1. Lebesgue differentiation theorem
2. Ergodic theorem for ergodic systems
3. Strong law of large numbers

Theorems satisfying Schnorr randomness

1. Lebesgue differentiation theorem
2. Ergodic theorem for ergodic systems
3. Strong law of large numbers

Constructive ⇒ Holds for Schnorr random

"Metatheorem"

Given a constructive almost-everywhere theorem (of say Bishop or Demuth):

(1) For all objects A, for almost-every x, it holds that $P(A, x)$.

Then the following (classically) holds:

(2) For all computable A and all Schnorr random x, it holds that $P(A, x)$.

"Proof."

A constructive proof of (1) must provide an algorithm constructing the (Schnorr) null set N from a code for A. □

Example: The ergodic theorem

- The **ergodic theorem** says for integrable f and measure preserving T,

$$A_n f(x) := \frac{1}{n}\sum_{k<n} f(T^n(x)) \quad \text{converges as } n \to \infty \qquad \text{for a.e. } x.$$

- T is **ergodic** if $g \circ T = g$ implies g is a.e. constant.

Theorem (Gács/Hoyrup/Rojas)

1. x is Schnorr random iff
2. $A_n f(x)$ converges for all "computable" f and "computable", ergodic T.

- "1 implies 2" also follows from known constructive results.
- Is ergodic the best "reasonable" assumption on the system? **No:**

Theorem (Spitters)

TFAE (constructively) for a fixed f and T:

- $A_n f$ converges almost-uniformly.
- The set of T-invariant functions $\{g \in L^1 : g \circ T = g\}$ is a located set in L^1.

Martin-Löf randomness and non-constructive results

Theorems characterizing Martin-Löf randomness

1. Lebesgue theorem on the differentiability of bounded variation functions
2. Ergodic theorem
3. Doob's martingale convergence theorem

- None of these theorems are constructive.
- They all implicitly compute the halting problem.
- They are also equivalent to ACA_0 in reverse math.

- However, some constructivity can be recovered.
- For example, Bishop [1967] has results which imply that Lebesgue's theorem holds for Martin-Löf randoms and computable b.v. functions.

Comments on the point-free approach

Point-free topologies

- Two approaches to working with the topological space $\mathbb{R}$:
  - Treat $\mathbb{R}$ as a **set of points** with a topological structure.
  - Treat $\mathbb{R}$ as a **lattice of open sets**.
- The second approach has advantages in constructive math.

- Two approaches to working with the measure space $([0,1],\mu)$.
  - Treat $[0,1]$ as a **set of points** with a measure structure.
  - Treat $([0,1],\mu)$ as a **lattice of measurable sets (mod null).**
- This lattice is a **locale**, i.e. it satisfies the axioms of a topology: has top and bottom and is closed under finite intersections and arbitrary unions.
- This **measurable locale** is an example of a point-free topology.

Randomness is "pointless"

- A point is **truely random** if it is in every measure one set.
- The measurable locale is the (point-free) space of true randoms.
- (This is Alex Simpson's idea, except he used the locale of Jordan-Peano measurable sets mod null.)
- Bas Spitters: "Randomness is pointless."

- Forcing over the measurable locale produces Solovay randoms.
- A **Solovay random** is in every measure one set in the ground model.

- Effectively forcing over the Boolean algebra of effectively measurable sets mod null produces Schnorr randoms.
- A Schnorr random is in every effective/constructive measure one set.

Closing Thoughts

Summary

- The early constructivists adopted the Brouwer/Schnorr definition of null sets because it has better properties.
- If an a.e. theorem is constructively provable, then it holds for Schnorr randomness.
- While there are Brouwer/Schnorr, Martin-Löf, and point-free approaches to measure theory, they are—for the most part—constructively inter-interpretable.
- The point-free approaches are elegant and allow one to avoid fiddly questions, however they are still connected to Schnorr randomness via effective versions of Solovay forcing.

Thank You!

These slides will be available on my webpage:

http://www.personal.psu.edu/jmr71/

Or just Google™ me, "Jason Rute".

P.S. This Autumn, I will be applying for jobs.