# Universality, optimality, and randomness deficiency

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# Randomness deficiency

## Definition (Martin-Löf)

Let $\vec{\mathcal{U}} = (\mathcal{U}_i)_{i \in \omega}$ be a universal ML-test. The randomness deficiency relative to $\vec{\mathcal{U}}$ of an $X \in \mathrm{MLR}$ is

$$\mathsf{RD}_{\vec{\mathcal{U}}}(X) = \min\{i : X \notin \mathcal{U}_i\}.$$

The idea is that the smaller $\mathsf{RD}_{\vec{\mathcal{U}}}(X)$ is, the more random $X$ is according to $\vec{\mathcal{U}}$.

# Layerwise computability

**Definition (Hoyrup & Rojas)**

Let $\vec{\mathcal{U}}$ be a universal ML-test. A function $F\colon 2^\omega \to 2^\omega$ is $\vec{\mathcal{U}}$-layerwise computable if there is a Turing functional $\Phi$ such that

$$\Phi(X, i) = F(X)$$

whenever $X \in 2^\omega \setminus \mathcal{U}_i$.

The idea is that $F(X)$ is uniformly computable on $\mathrm{MLR}$ if you're also given advice about the randomness deficiency of $X$.

This is a helpful notion for studying effectivity in Brownian motion, Birkhoff's ergodic theorem, convergence of random variables, etc. See also Pauly's talk at CCR.

# Weihrauch reducibility (suppressing representations)

"$F\colon \subseteq\omega^\omega \rightrightarrows \omega^\omega$" means that $F$ is a partial multi-valued function.

**Definition (Weihrauch)**

For $F, G\colon \subseteq\omega^\omega \rightrightarrows \omega^\omega$, $F \leq_{\mathrm{W}} G$ if there are Turing functionals $\Phi$ and $\Psi$ such that

$$\Psi(h, G(\Phi(h))) \subseteq F(h)$$

for all $h \in \mathrm{dom}(F)$.

That is, $\Psi(h, k) \in F(h)$ whenever $k \in G(\Phi(h))$.

- $\Phi$ takes $F$-inputs $h$ and processes them into $G$-inputs $\Phi(h)$.
- $\Psi$ takes $h$ and $G(\Phi(h))$-outputs $k$ and computes $F(h)$-outputs.

# A few notes on Weihrauch reducibility

Weihrauch reducibility generalizes to functions $F\colon \subseteq \mathcal{X} \rightrightarrows \mathcal{Y}$, where $\mathcal{X}$ and $\mathcal{Y}$ are, e.g., complete separable metric spaces. In this situation, we view elements of $\omega^\omega$ as coding elements of $\mathcal{X}$ and $\mathcal{Y}$.

However, today we mostly care about $2^\omega$ and $\omega$, so we ignore the details of such codings.

(View $2^\omega$ as a subspace of $\omega^\omega$, and identify $n \in \omega$ with $\{n\}$.)

$F \leq_{\mathrm{W}} G$ strengthens to $F \leq_{\mathrm{sW}} G$, where now $\Psi(G(\Phi(h))) \subseteq F(h)$ for all $h \in \mathrm{dom}(F)$.

In strong Weihrauch reducibility, the decoding function $\Psi$ does not have explicit access to $h$.

# A Weihrauch version of computing a function on $\mathrm{MLR}$ uniformly in the input's randomness deficiency

**Definition**

Let $\vec{\mathcal{U}}$ be a universal ML-test. Let $\mathsf{LAY}_{\vec{\mathcal{U}}} \colon \mathrm{MLR} \rightrightarrows \omega$ be defined by

$$\mathsf{LAY}_{\vec{\mathcal{U}}}(X) = \{i : X \notin \mathcal{U}_i\}.$$

$F \leq_{\mathrm{W}} \mathsf{LAY}_{\mathcal{U}}$ also expresses a sense in which $F$ is computable on $\mathrm{MLR}$ if you're given the ability to determine a random's randomness deficiency.

# What is this talk about?

**How do $\vec{\mathcal{U}}$-layerwise computability and Weihrauch reducibility to $\mathsf{LAY}_{\vec{\mathcal{U}}}$ compare?**

- Both express similar ideas: uniform computability on $\mathrm{MLR}$ given randomness deficiencies.
- The pre-processing power of $\Phi$ in the definition of Weihrauch reducibility makes Weihrauch reducibility to $\mathsf{LAY}_{\vec{\mathcal{U}}}$ more powerful than $\vec{\mathcal{U}}$-layerwise computability.

**Does the choice of $\vec{\mathcal{U}}$ matter?**

- For $\vec{\mathcal{U}}$-layerwise computability it matters, but you have to make a purposefully stupid choice of $\vec{\mathcal{U}}$.
- For Weihrauch reducibility to $\mathsf{LAY}_{\vec{\mathcal{U}}}$, it doesn't matter.

# Purposefully stupid = universal but not optimal

**Definition**

Let $\vec{\mathcal{U}}$ be an ML-test.

- $\vec{\mathcal{U}}$ is universal if $\bigcap_{i \in \omega} \mathcal{U}_i = 2^\omega \setminus \mathrm{MLR}$.
- $\vec{\mathcal{U}}$ is optimal if for every ML-test $\vec{\mathcal{V}}$ there is a $c$ such that $\forall i (\mathcal{V}_{i+c} \subseteq \mathcal{U}_i)$.

Every optimal ML-test is universal, and there are optimal ML-tests.

There are universal ML-tests that are not optimal.

# A nice difference between universal and optimal ML-tests

**Theorem (Merkle, Mihailović, Slaman)**

*There are a* **universal** *ML-test* $\vec{\mathcal{U}}$ *and a left-r.e. real* $\alpha$ *such that*

$$\forall i(\lambda(\mathcal{U}_i) = 2^{-i}\alpha).$$

**Theorem (Miyabe)**

*No* **optimal** *ML-test can witness the previous theorem.*

# Optimal tests and layerwise computability

Recall that $F$ is $\vec{\mathcal{U}}$-layerwise computable if there is a Turing functional $\Phi$ such that $F(X) = \Phi(X, i)$ whenever $X \in 2^\omega \setminus \mathcal{U}_i$.

Hoyrup & Rojas only defined $\vec{\mathcal{U}}$-layerwise computability for **optimal** tests.

It is easy to check that if $\vec{\mathcal{U}}$ and $\vec{\mathcal{V}}$ are universal ML-tests and $f\colon \omega \to \omega$ is a recursive function such that

$$\forall i(\mathcal{V}_{f(i)} \subseteq \mathcal{U}_i)$$

then every $\vec{\mathcal{V}}$-layerwise computable function is $\vec{\mathcal{U}}$-layerwise computable.

Hence optimal ML-tests give the most general notion of layerwise computability.

How badly non-optimal can a universal ML-test be?

# Badly non-optimal universal ML-tests

If $\vec{\mathcal{U}}$ and $\vec{\mathcal{V}}$ are universal ML-tests, must there be an $f\colon \omega \to \omega$ such that $\forall i(\mathcal{V}_{f(i)} \subseteq \mathcal{U}_i)$? (That is, must it be that $\forall i \exists j(\mathcal{V}_j \subseteq \mathcal{U}_i)$?)

If there is such an $f$, how hard is it to compute? (If there is an $f$, then there is an $f \leq_{\mathrm{T}} 0''$.)

**Theorem (H&S)**

*There are universal ML-tests $\vec{\mathcal{U}}$ and $\vec{\mathcal{V}}$ such that $\exists i \forall j(\mathcal{V}_j \not\subseteq \mathcal{U}_i)$.*

**Theorem (H&S)**

*There is a universal ML-test $\vec{\mathcal{U}}$ such that*

- *if $\vec{\mathcal{V}}$ is any ML-test, then $\forall i \exists j(\mathcal{V}_j \subseteq \mathcal{U}_i)$ and*
- *if $\vec{\mathcal{V}}$ is any optimal ML-test and $f\colon \omega \to \omega$ is such that $\forall i(\mathcal{V}_{f(i)} \subseteq \mathcal{U}_i)$, then $f \geq_{\mathrm{T}} 0''$.*

# Layerwise computability depends on the test

By the previous slide, there are universal ML-tests for which no computable function (or any function) can translate between the layerings.

**Theorem (H&S)**

*There are universal ML-tests $\vec{\mathcal{U}}$ and $\vec{\mathcal{V}}$ and a function $F$ such that $F$ is $\vec{\mathcal{U}}$-layerwise computable but* **not** *$\vec{\mathcal{V}}$-layerwise computable.*

- $\mathcal{A} \subseteq 2^\omega$ is effectively measurable if there are uniformly r.e. sequences of open sets $\vec{\mathcal{O}}$, $\vec{\mathcal{C}}$ such that $2^\omega \setminus \mathcal{C}_i \subseteq \mathcal{A} \subseteq \mathcal{O}_i$ and $\lambda(\mathcal{O}_i \cap \mathcal{C}_i) \leq 2^{-i}$ for all $i \in \omega$.
- (Hoyrup & Rojas) For an optimal ML-test $\vec{\mathcal{U}}$, a set is effectively measurable if and only if its characteristic function is $\vec{\mathcal{U}}$-layerwise computable.
- There is an effectively measurable set $\mathcal{A}$ and universal ML-test $\vec{\mathcal{V}}$ such that the characteristic function of $\mathcal{A}$ is **not** $\vec{\mathcal{V}}$-layerwise computable.

# Weihrauch reducibility to LAY does not depend on the test

Let $\vec{\mathcal{U}}$ be a universal ML-test. Recall that for $X \in \mathrm{MLR}$

- $\mathsf{LAY}_{\vec{\mathcal{U}}}(X) = \{i : X \notin \mathcal{U}_i\}$ and
- $\mathsf{RD}_{\vec{\mathcal{U}}}(X) = \min\{i : X \notin \mathcal{U}_i\}$.

**Theorem (H&S)**

$\mathsf{LAY}_{\vec{\mathcal{U}}} \equiv_{\mathrm{W}} \mathsf{RD}_{\vec{\mathcal{V}}}$ *for every pair of universal ML-tests* $\vec{\mathcal{U}}$ *and* $\vec{\mathcal{V}}$.

(This theorem and many others concerning the Weihrauch degrees was proved independently by Pauly, Davie, and Fouché.)

So we may unambiguously refer to this Weihrauch degree as ‘LAY.’

# $\mathsf{LAY}_{\vec{\mathcal{U}}} \equiv_{\mathrm{W}} \mathsf{RD}_{\vec{\mathcal{V}}}$

**Theorem (H&S)**

$\mathsf{LAY}_{\vec{\mathcal{U}}} \equiv_{\mathrm{W}} \mathsf{RD}_{\vec{\mathcal{V}}}$ *for every pair of universal ML-tests* $\vec{\mathcal{U}}$ *and* $\vec{\mathcal{V}}$.

The interesting direction is $\mathsf{RD}_{\vec{\mathcal{V}}} \leq_{\mathrm{W}} \mathsf{LAY}_{\vec{\mathcal{U}}}$.

**Plan**: Given $X \in \mathrm{MLR}$, inflate $\mathsf{RD}_{\vec{\mathcal{U}}}(X)$ until it witnesses $\mathsf{RD}_{\vec{\mathcal{V}}}(X)$.

- $\Phi(X)$ copies $X$ while searching for $s_0$ such that $X \in \mathcal{V}_{0,s_0}$.
- If found, $\Phi(X)$ takes its current output $\sigma$, searches for $\tau$ such that $[\sigma^\frown\tau] \subseteq \bigcap_{i \leq s} \mathcal{U}_i$, and appends $\tau$ to its output.
- $\Phi$ resumes copying $X$ while searching for $s_1$ such that $X \in \mathcal{V}_{1,s_1}$ ....
- In the end, $\Phi(X) \in \mathrm{MLR}$ is such that $i < \mathsf{RD}_{\vec{\mathcal{V}}}(X) \Rightarrow \Phi(X) \in \mathcal{V}_{i,\mathsf{RD}_{\vec{\mathcal{U}}}(X)}$.
- Let $\Psi(X, k)$ be the least $i$ such that $X \notin \mathcal{V}_{i,k}$.

# What about strong Weihrauch reducibility?

In the proof of $\mathsf{RD}_{\vec{\mathcal{V}}} \leq_{\mathrm{W}} \mathsf{LAY}_{\vec{\mathcal{U}}}$, the decoding function $\Psi(X, k)$ made essential use of $X$.

The theorem cannot be improved to $\leq_{\mathrm{sW}}$.

However, the $\mathsf{LAY}_{\vec{\mathcal{U}}}$ are all equivalent up to strong Weihrauch degree.

**Proposition (H&S)**

*Let $\vec{\mathcal{U}}$ and $\vec{\mathcal{V}}$ be universal ML-tests. Then*

- $\mathsf{RD}_{\vec{\mathcal{V}}} \not\leq_{\mathrm{sW}} \mathsf{LAY}_{\vec{\mathcal{U}}}$ *and*
- $\mathsf{LAY}_{\vec{\mathcal{U}}} \equiv_{\mathrm{sW}} \mathsf{LAY}_{\vec{\mathcal{V}}}$.

**Question**: Must $\mathsf{RD}_{\vec{\mathcal{U}}} \equiv_{\mathrm{sW}} \mathsf{RD}_{\vec{\mathcal{V}}}$?

# Layerwise computability vs. Weihrauch reducibility to LAY

Let $\vec{\mathcal{U}}$ be a universal ML-test.

It is easy to check that if $F$ is $\vec{\mathcal{U}}$-layerwise computable, then $F \restriction \mathrm{MLR} \leq_{\mathrm{W}}$ LAY.

An obvious question: Is $\mathsf{RD}_{\vec{\mathcal{U}}}$ a $\vec{\mathcal{U}}$-layerwise computable function?

**Theorem (H&S)**

*Let $\vec{\mathcal{U}}$ be a universal ML-test. Then $\mathsf{RD}_{\vec{\mathcal{U}}}$ is* **not** *$\vec{\mathcal{U}}$-layerwise computable.*

We know that $\mathsf{RD}_{\vec{\mathcal{U}}} \leq_{\mathrm{W}}$ LAY, so $\mathsf{RD}_{\vec{\mathcal{U}}}$ is an example of a function that is Weihrauch reducible to LAY but not $\vec{\mathcal{U}}$-layerwise computable.

# Layerwise semi-decidability

**Definition (Hoyrup & Rojas)**

Let $\vec{\mathcal{U}}$ be a universal ML-test.

- $\mathcal{A} \subseteq 2^\omega$ is $\vec{\mathcal{U}}$-layerwise semi-decidable if there is a uniformly r.e. sequence of open sets $\vec{\mathcal{O}}$ such that

$$\forall i[\mathcal{A} \cap (2^\omega \setminus \mathcal{U}_i) = \mathcal{O}_i \cap (2^\omega \setminus \mathcal{U}_i)].$$

- $\mathcal{A} \subseteq 2^\omega$ is $\vec{\mathcal{U}}$-layerwise decidable if $\mathcal{A}$ and $2^\omega \setminus \mathcal{A}$ are $\vec{\mathcal{U}}$-layerwise semi-decidable.

Easy to check that $\mathcal{A}$ is $\vec{\mathcal{U}}$-layerwise decidable if and only if its characteristic function is $\vec{\mathcal{U}}$-layerwise computable.

# Layerwise semi-decidability vs. Weihrauch reducibility

The characteristic function of every layerwise semi-decidable set Weihrauch reduces to LAY:

**Theorem (H&S)**

*If $\vec{\mathcal{U}}$ is a universal ML-test and $\mathcal{A} \subseteq 2^\omega$ is $\vec{\mathcal{U}}$-layerwise semi-decidable, then $\chi_{\mathcal{A}} \restriction \mathrm{MLR} \leq_{\mathrm{W}}$ LAY.*

**Proposition (Hoyrup & Rojas)**

*Let $\vec{\mathcal{U}}$ be a universal ML-test, and let $\mathcal{A}$ be $\vec{\mathcal{U}}$-layerwise semi-decidable. Then $\mathcal{A}$ is $\vec{\mathcal{U}}$-layerwise decidable if and only if $\lambda(\mathcal{A})$ is recursive.*

So there are lots of functions that Weihrauch reduce to LAY but are not layerwise computable.

# Exact layerwise computability

We have seen that $\mathsf{RD}_{\vec{\mathcal{U}}} \equiv_{\mathrm{W}} \mathsf{LAY}_{\vec{\mathcal{U}}}$ for any universal ML-test $\vec{\mathcal{U}}$.

Thus up to Weihrauch degree, producing $\mathsf{RD}_{\vec{\mathcal{U}}}(X)$ for an $X \in \mathrm{MLR}$ is equivalent to producing an upper bound for $\mathsf{RD}_{\vec{\mathcal{U}}}(X)$.

What if we strengthen the definition of $\vec{\mathcal{U}}$-layerwise computability to require the exact value of $\mathsf{RD}_{\vec{\mathcal{U}}}(X)$?

**Definition**

Let $\vec{\mathcal{U}}$ be a universal ML-test. A function $F\colon 2^\omega \to 2^\omega$ is exactly $\vec{\mathcal{U}}$-layerwise computable if there is a Turing functional $\Phi$ such that $\Phi(X, \mathsf{RD}_{\vec{\mathcal{U}}}(X)) = F(X)$ for every $X \in \mathrm{MLR}$.

# Exact layerwise computability vs. layerwise computability

Let $\vec{\mathcal{U}}$ be a universal ML-test.

Clearly $\mathrm{RD}_{\vec{\mathcal{U}}}$ is exactly $\vec{\mathcal{U}}$-layerwise computable.

But we have seen that $\mathrm{RD}_{\vec{\mathcal{U}}}$ is not $\vec{\mathcal{U}}$-layerwise computable.

So there are functions that are exactly layerwise computable but not layerwise computable.

Also, exact layerwise computability depends on the test.

**Theorem (H&S)**

*There are universal ML-tests $\vec{\mathcal{U}}$ and $\vec{\mathcal{V}}$ and a function $F$ such that $F$ is exactly $\vec{\mathcal{U}}$-layerwise computable but* **not** *exactly $\vec{\mathcal{V}}$-layerwise computable.*

# Exact layerwise computability vs. Weihrauch reducibility

Let $\vec{\mathcal{U}}$ be a universal ML-test.

If $F\colon 2^\omega \to 2^\omega$ is exactly $\mathcal{U}$-layerwise computable, then $F \restriction \mathrm{MLR} \leq_{\mathrm{W}} \mathsf{LAY}$.

This is essentially because $\mathsf{RD}_{\vec{\mathcal{U}}} \equiv_{\mathrm{W}} \mathsf{LAY}$.

Still, there are functions Weihrauch reducible to LAY that are not exactly $\vec{\mathcal{U}}$-layerwise computable.

**Theorem (H&S)**

*Let $\vec{\mathcal{U}}$ be a universal ML-test. Then there is a function $F \leq_{\mathrm{W}} \mathsf{LAY}$ that is not exactly $\vec{\mathcal{U}}$-layerwise computable.*

# Algebraic operations in the Weihrauch degrees

Let $f$ and $g$ be partial multi-valued functions. Define

- $(f \times g)(x, y) = f(x) \times g(y)$ and
- $(f * g)(x) = \max\{f_0 \circ g_0 : (f_0 \leq_{\mathrm{W}} f) \wedge (g_0 \leq_{\mathrm{W}} g)\}$ (always exists by Brattka & Pauly).

Additionally, consider the following two functions:

- For $\mathcal{A} \subseteq \omega^\omega$, $\mathrm{id}_\mathcal{A}$ is the identity function but with domain restricted to $\mathcal{A}$.
- $\mathrm{C}_\mathbb{N} \colon \subseteq \omega^\omega \rightrightarrows \omega$ is the multi-valued function with domain

$$\{f \in \omega^\omega : \exists n \forall k (f(k) \neq n + 1)\}$$

defined by

$$\mathrm{C}_\mathbb{N}(f) = \omega \setminus \{n : \exists k (f(k) \neq n + 1)\}.$$

# Algebraic properties of LAY in the Weihrauch degrees

**Theorem (H&S; Pauly, Davie, and Fouché)**

$\mathsf{LAY} * \mathsf{LAY} \equiv_{\mathrm{W}} \mathsf{LAY}$

It follows that $\mathsf{LAY} \times \mathsf{LAY} \equiv_{\mathrm{W}} \mathsf{LAY}$ as well. This can be improved to $\mathsf{LAY} \times \mathsf{LAY} \equiv_{\mathrm{sW}} \mathsf{LAY}$.

**Theorem (H&S)**

- $\mathsf{LAY} \leq_{\mathrm{sW}} \mathrm{C}_{\mathbb{N}}$
- $\mathrm{C}_{\mathbb{N}} \not\leq_{\mathrm{W}} \mathsf{LAY}$ (also Pauly, Davie, and Fouché)
- $\mathsf{LAY} \equiv_{\mathrm{W}} \mathrm{C}_{\mathbb{N}} \times \mathrm{id}_{\mathrm{MLR}}$ (also Pauly, Davie, and Fouché)

# Danke!

Thank you for coming to my talk!
Do you have a question about it?