Constructive dimension and Hausdorff dimension: the case of exact dimension

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Outline

Notation: Strings and languages

Finite Alphabet $X = \{0, \ldots, 2-1\}$, cardinality $|\{0,1\}| = 2$

Finite strings (words) $w = x_1 \cdots x_n \in \{0,1\}^*$, $x_i \in \{0,1\}$

Length $|w| = n$

Languages $V, W \subseteq \{0,1\}^*$

Infinite strings (ω-words) $\xi = x_1 \cdots x_n \cdots \in \{0,1\}^\omega$

Prefixes of infinite strings $\xi[0..n] \in \{0,1\}^*$, $\left|\xi[0..n]\right| = n$

ω-Languages $F \subseteq \{0,1\}^\omega$

$\{0,1\}^\omega$ as CANTOR space

Metric: $\rho(\eta,\xi) := \inf\{2^{-|w|} : w \in \mathbf{pref}(\eta) \cap \mathbf{pref}(\xi)\}$

Balls: $w \cdot \{0,1\}^\omega = \{\eta : w \in \mathbf{pref}(\eta)\}$

Diameter: $\operatorname{diam} w \cdot \{0,1\}^\omega = 2^{-|w|}$

$\operatorname{diam} F = \inf\{2^{-|w|} : F \subseteq w \cdot \{0,1\}^\omega\}$

Open sets: $W \cdot \{0,1\}^\omega = \bigcup_{w \in W} w \cdot \{0,1\}^\omega$

Closure: $\mathscr{C}(F) = \{\xi : \mathbf{pref}(\xi) \subseteq \mathbf{pref}(F)\}$

Gambling strategies

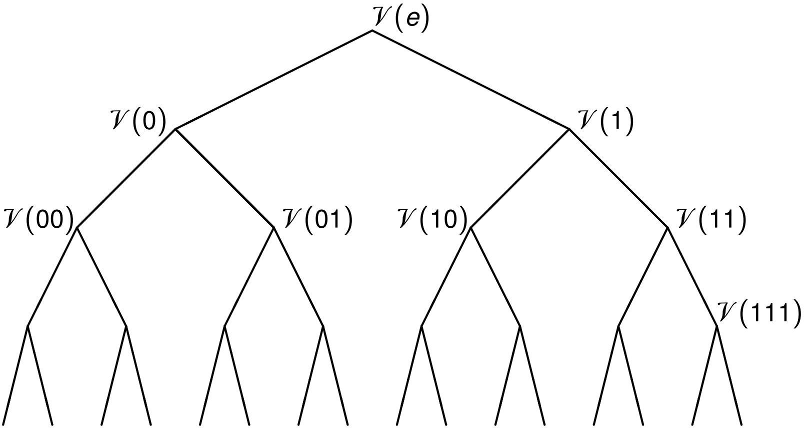
Our model:

- Playing head-and-tails against a binary sequence $\xi \in \{0,1\}^\omega$
- Gambling strategy $\Gamma : \{0,1\}^* \to [0,1]$ (bet on outcome 1)
- yields a (super-)martingale $\mathcal{V}_\Gamma : \{0,1\}^* \to \mathbb{R}_+$
- $\mathcal{V}_\Gamma(\xi[0..n])$ is the capital after the n the round

Fact (super-martingale property)

$$\mathcal{V}_\Gamma(w) \geq \sum_{x \in \{0,1\}} \frac{1}{2} \cdot \mathcal{V}_\Gamma(wx)$$

Gambling strategies: martingale $\mathcal{V}$



How much can you win: Order functions [SCHNORR'71]

Definition (Order function f and gauge function h)

$f : \mathbb{N} \to \mathbb{N}$ increasing

$h : (0,\infty) \to (0,\infty)$ right continuous and increasing

gauge function		order function
$h : (0,\infty) \to (0,\infty)$		$f : \mathbb{N} \to \mathbb{N}$
$h(2^{-n}) = 2^{-n} \cdot f(n)$	$\longleftrightarrow$	$f(n)$

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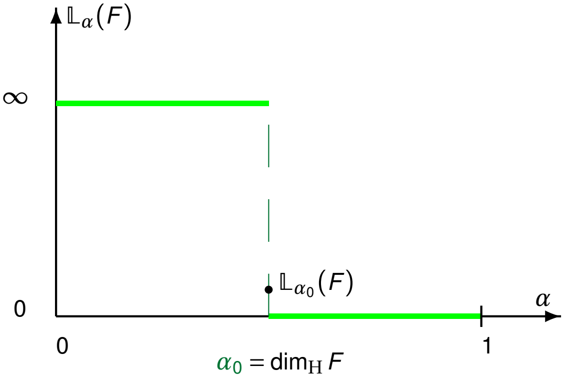
Definition (success set)

$$S_{c,h}[\mathcal{V}] := \left\{ \xi : \xi \in \{0,1\}^{\omega} \wedge \limsup_{n\to\infty} \frac{\mathcal{V}(\xi[0..n])}{2^n \cdot h(2^{-n})} \geq c \right\},\ c \in (0,\infty) \cup \{\infty\}$$

“Classical” HAUSDORFF dimension

HAUSDORFF dimension of $F \subseteq \{0,1\}^\omega$

$$\mathbb{L}_\alpha(F) := \lim_{n\to\infty} \inf\Big\{ \sum_{v\in V} 2^{-\alpha\cdot|v|} : F \subseteq \bigcup_{v\in V} v\cdot\{0,1\}^\omega \wedge \min_{v\in V} |v| \geq n \Big\}$$



Relation to HAUSDORFF dimension

Let $S_{c,\alpha}[\mathcal{V}] := \left\{\xi : \limsup_{n\to\infty} \frac{\mathcal{V}(\xi[0..n])}{2^n \cdot 2^{-\alpha\cdot n}} \geq c\right\}$ for $c \in (0,\infty) \cup \{\infty\}$

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Lemma

For every super-martingale $\mathcal{V}$:

$$\dim_{\mathrm{H}} S_{c,\alpha}[\mathcal{V}] \leq \alpha$$

Theorem ([LUTZ'03])

Let $F \subseteq \{0,1\}^\omega$. Then

$$\dim_{\mathrm{H}} F < \alpha \;\rightarrow\; \exists \mathcal{V}(F \subseteq S_{\infty,\alpha}[\mathcal{V}]) \;\rightarrow\; \dim_{\mathrm{H}} F \leq \alpha\,.$$

Semi-computability

Definition (Left computable)

A (partial) mapping $\phi : \{0,1\}^* \to \mathbb{R}$ is called **computably approximable from below** (*left computable*) : $\iff$

$$\{(w,q) : w \in \operatorname{dom}\phi \wedge q \in \mathbb{Q} \wedge q < \phi(w)\}$$

is computably enumerable.

Similar: **computably approximable from above** (*right computable*)

Universal super-martingale

Definition (Continuous (or: Cylindrical) semi-measure)

$$\mu(w) \geq \mu(w0) + \mu(w1)$$

Theorem (LEVIN'70)

There is a universal left computable continuous semi-measure $\mathbf{M}$ *on* $\{0,1\}^*$, *that is,* $\mathbf{M}$ *is left computable and for every left computable continuous semi-measure* μ *there is a constant* c_μ *such that*

$$\mu(w) \leq c_\mu \cdot \mathbf{M}(w) \quad \textit{for all } w \in \{0,1\}^* .$$

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Theorem (Levin'70, Schnorr'71)

There is a universal left computable super-martingale $\mathcal{U}$*, e.g.*

$$\mathcal{U}(w) := 2^{|w|} \cdot \mathbf{M}(w).$$

Constructive dimension [LUTZ'03]

Constructive dimension tries to measure for $\xi \in \{0,1\}^\omega$ the exponent α for which

$$\mathscr{U}(\xi[0..n]) \approx 2^{\alpha \cdot n + o(n)}.$$

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More precisely,

$$\begin{array}{llll} \mathscr{U}(\xi[0..n]) & \geq_{i.o.} & 2^{\alpha' \cdot n} & \text{for } \alpha' < \alpha \text{ , and} \\ \mathscr{U}(\xi[0..n]) & \leq_{a.e.} & 2^{\alpha'' \cdot n} & \text{for } \alpha'' > \alpha \, , \end{array}$$

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that is [LEVIN'70, LUTZ'03],

Definition (Constructive dimension of ξ)

$$\underline{\kappa}(\xi) := \quad 1 - \alpha = \liminf_{n \to \infty} \frac{-\log \mathbf{M}(\xi[0..n])}{n}$$

Corollary (LUTZ'03)

Let $F \subseteq \{0,1\}^\omega$. Then

$$\sup\{\underline{\kappa}(\xi) : \xi \in F\} = \inf\left\{\alpha : F \subseteq S_{\infty,\alpha}[\mathscr{U}]\right\}.$$

KOLMOGOROV complexity – USPENSKY-SHEN-pentagon

***a priori* complexity** $\mathrm{KA}(w) := -\log \mathbf{M}(w)$

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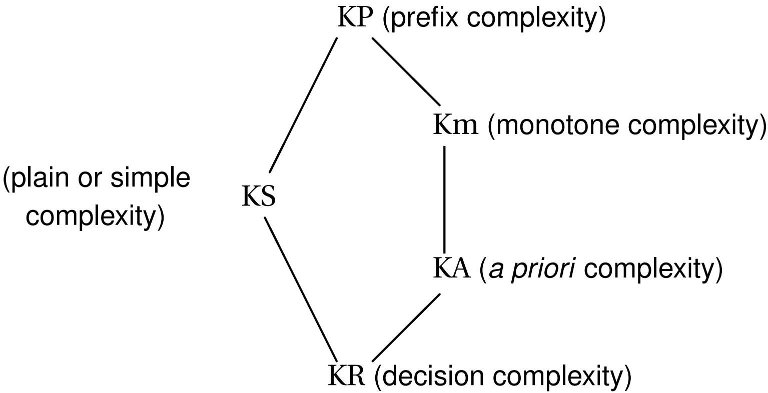

The original approach [HAUSDORFF '18]

Classical

$$\mathbb{L}_\alpha(F) := \lim_{n\to\infty} \inf\Big\{\sum_{v\in V} (2^{-|v|})^{\alpha} : F \subseteq \bigcup_{v\in V} v\cdot\{0,1\}^\omega \wedge \min_{v\in V}|v| \geq n\Big\}$$

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Original

$$\mathscr{H}^h(F) := \lim_{n\to\infty} \inf\left\{ \sum_{v\in V} h(2^{-|v|}) : F \subseteq \bigcup_{v\in V} v\cdot\{0,1\}^\omega \wedge \min_{v\in V} |v| \geq n \right\}$$

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where h is a *gauge function*, that is,
$h:(0,\infty)\to(0,\infty)$ is right continuous and non-decreasing,

The gauge functions for the "classical" HAUSDORFF dimension are $h_\alpha(t) = t^\alpha$.

Exact HAUSDORFF dimension I

Lemma ([HAUSDORFF'18])

Let $h, h' : \mathbb{R}_+ \to \mathbb{R}_+$ be gauge functions and let $F \subseteq \{0,1\}^\omega$.

1. *If $h(t) \leq c \cdot h'(t)$ then $\mathscr{H}^h(F) \leq c \cdot \mathscr{H}^{h'}(F)$.*
2. *If $\lim_{t \to 0} \frac{h(t)}{h'(t)} = 0$ then $\mathscr{H}^{h'}(F) < \infty$ implies $\mathscr{H}^h(F) = 0$, and $\mathscr{H}^h(F) > 0$ implies $\mathscr{H}^{h'}(F) = \infty$.*

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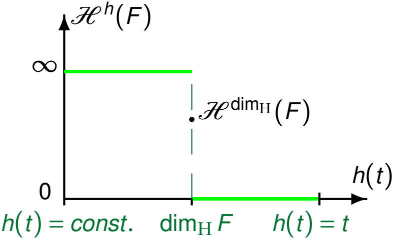
Quasi-ordering of gauge functions (Speed of converging to 0)

Largest	$h(t) = t$
Ordering	$h(t') \prec h(t)$ if and only if $\lim_{t\to 0} \frac{h(t)}{h'(t)} = 0$,
	e.g. the exponential functions $h(t) = t^\alpha, 0 \leq \alpha \leq 1$
Smallest	$h(t) = const. > 0$

Exact HAUSDORFF dimension II



Definition (Exact HAUSDORFF dimension)

We refer to a gauge function h as *an* **exact Hausdorff dimension function** for $F \subseteq \{0,1\}^\omega$ provided

$$\mathscr{H}^{h'}(F) = \begin{cases} \infty, & \text{if } \lim_{t\to 0} \frac{h(t)}{h'(t)} = 0 \text{ , and} \\ 0, & \text{if } \lim_{t\to 0} \frac{h'(t)}{h(t)} = 0 . \end{cases}$$

Martingale characterisation of exact HAUSDORFF dimension

Definition (Success set)

$$S_{c,h}[\mathcal{V}] := \left\{\xi : \limsup_{n\to\infty} \frac{\mathcal{V}(\xi[0..n])}{2^n \cdot h(2^{-n})} \geq c\right\} \text{ for } c \in (0,\infty) \cup \{\infty\}$$

Theorem

h is an exact Hausdorff dimension function for $F \subseteq \{0,1\}^\omega :\Longleftrightarrow$

1. *for all gauge functions h' with $\lim_{t\to 0} \frac{h'(t)}{h(t)} = 0$ there is a (super-)martingale $\mathcal{V}$ such that $F \subseteq S_{\infty,h'}[\mathcal{V}]$, and*
2. *$F \nsubseteq S_{\infty,h''}[\mathcal{V}]$ for all (super-)martingales $\mathcal{V}$ and all gauge functions h'' with $\lim_{t\to 0} \frac{h(t)}{h''(t)} = 0$.*

Exact constructive dimension

Analogously to the martingale characterisation of the Hausdorff dimension we set:

Definition

We refer to a gauge function h as an **exact constructive dimension function** for $F \subseteq \{0,1\}^\omega$ provided

1. $F \subseteq S_{\infty,h'}[\mathscr{U}]$ for all gauge functions h' with $\lim_{t\to 0} \frac{h'(t)}{h(t)} = 0$, and
2. $F \not\subseteq S_{\infty,h''}[\mathscr{U}]$ for all gauge functions h'' with $\lim_{t\to 0} \frac{h(t)}{h''(t)} = 0$.

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Theorem (Exact dimension for $\{\xi\}$)

The function h_ξ defined by $h_\xi(2^{-n}) := 2^{-n} \cdot \mathscr{U}(\xi[0..n]) = \mathbf{M}(\xi[0..n])$ is an exact constructive dimension function for the set $\{\xi\}$.

RYABKO's result: Large sets contain complex ω-words

Theorem (RYABKO'84, "classical case")

For $\alpha \in [0,1]$ *it holds* $\alpha = \dim_{\mathrm{H}}\{\xi : \xi \in \{0,1\}^\omega \wedge \underline{\kappa}(\xi) \leq \alpha\}$.

Theorem (*St'93*, "classical case")

If $F \subseteq \{0,1\}^\omega$ *and* $\mathbb{L}_\alpha(F) > 0$ *then there is a* $\xi \in F$ *such that*

$$\liminf_{n\to\infty} \frac{\mathrm{KS}(\xi[0..n])}{n} \geq_{\mathrm{a.e.}} \alpha \cdot n - (1+\varepsilon)\log n.$$

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Theorem (Lower KA-bound, Mielke'09)

Let $F \subseteq \{0,1\}^\omega$, h be a gauge function and $\mathcal{H}^h(F) > 0$.
Then for every $c > 0$ with $\mathcal{H}^h(F) > c \cdot \mathbf{M}(e)$ there is a $\xi \in F$ such that

$$\mathrm{KA}(\xi[0..n]) \geq_{\mathrm{a.e.}} -\log h(2^{-n}) + c.$$

Complexity bounds

Fact

$$\{\xi : \exists c(\mathrm{KA}(\xi[0..n]) \leq_{\text{i.o.}} -\log h(2^{-n}) + c)\} = \bigcup_{c' \in (0,\infty)} S_{c',h}[\mathcal{U}]$$

Corollary (to Lower KA-bound)

Let h, h' be gauge functions such that $\lim_{t \to 0} \frac{h'(t)}{h(t)} = 0$. Then

1. $\{\xi : \exists c(\mathrm{KA}(\xi[0..n]) \leq_{\text{i.o.}} -\log h(2^{-n}) + c)\} \subseteq S_{\infty,h'}[\mathcal{U}]$, *and*
2. $\mathcal{H}^{h'}(\{\xi : \exists c(\mathrm{KA}(\xi[0..n]) \leq_{\text{i.o.}} -\log h(2^{-n}) + c)\}) = 0$.

Exact dimension function: an example

Example

$$F := \left\{ x_1 x_2 \cdots x_i \cdots \,\middle|\, x_i \in \{0,1\} \wedge \forall j \left(x_{2^j} = 0 \right) \right\}$$

F has classical Hausdorff dimension $\dim_{\mathrm{H}} F = 1$ but does not contain any random sequence.

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The exact Hausdorff dimension is

$$\dim_{\mathrm{H}} F = \left[h(t) = t \cdot \log \tfrac{1}{t} \right].$$

Observe $-\log h(2^{-n}) = n - \log n$.

Upper bounds: "Classical case" [*St'98*]

Definition (Σ_2-sets)

A subset $F \subseteq \{0,1\}^\omega$ is Σ_2-definable if there is a computable set $W \subseteq \mathbb{N} \times \{0,1\}^*$ such that

$$\xi \in F \longleftrightarrow \exists m \forall n \big((m, \xi[0..n]) \in W\big).$$

Theorem

If $F \subseteq \{0,1\}^\omega$ is Σ_2-definable and $\alpha \geq \dim_{\mathrm{H}} F$ is a right computable real then there is a computable $V \subseteq \{0,1\}^*$ such that $F \subseteq V^\delta$ and $\sum_{v \in V} 2^{-|v|} < \infty$.

Corollary

If $F_i \subseteq \{0,1\}^\omega$, $i \in \mathbb{N}$, are Σ_2-definable then $\underline{\kappa}(\xi) \leq \dim \bigcup_{i \in \mathbb{N}} F_i$ for $\xi \in \bigcup_{i \in \mathbb{N}} F_i$.

Upper bound on prefix complexity KP

Lemma (Reimann'04)

Let $F \subseteq \{0,1\}^\omega$ and h be a gauge function. Then $\mathscr{H}^h(F) = 0$ if and only if there is a $V \subseteq \{0,1\}^*$ such that $F \subseteq V^\delta := \{\xi : |\mathbf{pref}(\xi) \cap V| = \infty\}$ and $\sum_{v \in V} h(2^{-|v|}) < \infty$.

Theorem

If $F \subseteq \{0,1\}^\omega$ is Σ_2-definable and h is a right computable gauge function such that $\mathscr{H}^h(F) = 0$ then there are a non-decreasing function $\overline{h} : \{2^{-i} : i \in \mathbb{N}\} \to \mathbb{Q}$ and a computable $V \subseteq \{0,1\}^*$ such that

1. $\overline{h}(2^{-i}) \geq h(2^{-i})$ *for* $i \in \mathbb{N}$,
2. $\sum_{v \in V} \overline{h}(2^{-|v|}) < \infty$ *and* $F \subseteq V^\delta$, *and*
3. $\mathrm{KP}(\xi[0..n]) \leq_{\mathrm{i.o.}} -\log_r h(r^{-n}) + O(1)$ *for all* $\xi \in F$.

Upper bounds: Computable dimension

Theorem ("Classical case", *St'98*)

If $F \subseteq \{0,1\}^\omega$ is Σ_2-definable and $\alpha \geq \dim_{\mathrm{H}} F$ is a right computable real then there is a computable martingale $\mathcal{V}$ such that $F \subseteq S_{\infty,\alpha}[\mathcal{V}]$.

Theorem

For every Σ_2-definable $F \subseteq \{0,1\}^\omega$ and every computable gauge function $h : \mathbb{Q} \to \mathbb{R}$ such that $\mathscr{H}^h(F) = 0$ there is a computable martingale $\mathcal{V}$ such that $F \subseteq \bigcup_{c>0} S_{c,h}[\mathcal{V}]$.

Dilution functions

Modulus function: $g : \mathbb{N} \to \mathbb{N}$ strictly monotone, that is, $g(n+1) > g(n)$

Example [Dilution function with $|\varphi(w)| = g(|w|)$]

$\varphi : \{0,1\}^* \to \{0,1\}^*$

$$\begin{aligned} \varphi(e) &:= 0^{g(0)} \text{ and} \\ \varphi(wx) &:= \varphi(w) \cdot x \cdot 0^{g(n+1)-g(n)-1} \end{aligned}$$

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Definition (Dilution function)

For every $v \in \mathbf{pref}(\varphi(\{0,1\}^*))$ there are $w_v \in \{0,1\}^*$ and $x_v \in \{0,1\}$ such that

$$\varphi(w_v) \sqsubset v \sqsubseteq \varphi(w_v \cdot x_v) \quad \wedge \quad \forall y\big(y \in \{0,1\} \wedge y \neq x_v \to v \not\sqsubseteq \varphi(w_v \cdot y)\big)$$

Hausdorff measure of diluted sets

Theorem

Let $g:\mathbb{N}\to\mathbb{N}$ be a strictly increasing function, φ a corresponding dilution function and $h:(0,\infty)\to(0,\infty)$ be a gauge function. Then

1. $\mathscr{H}^h(\overline{\varphi}(\{0,1\}^\omega)) \leq \liminf_{n\to\infty} \frac{h(2^{-g(n)})}{2^{-n}}$, *and*
2. *if* $c\cdot 2^{-n} \leq_{\text{a.e.}} h(2^{-g(n)})$ *then* $c \leq \mathscr{H}^h(\overline{\varphi}(\{0,1\}^\omega))$.

Corollary

If $c\cdot 2^{-n} \leq_{\text{a.e.}} h(2^{-g(n)}) \leq c'\cdot 2^{-n}$ *then* $c \leq \mathscr{H}^h(\overline{\varphi}(\{0,1\}^\omega)) \leq c'$.

Dilution: an existence condition for modulus functions

Lemma (Sufficient condition)

If a gauge function h is upwardly convex (or $\cap$-convex) on some interval $(0,\varepsilon)$ and $\lim_{t\to 0} h(t) = 0$ then there is an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ there is an $m \in \mathbb{N}$ satisfying

$$2^{-n} < h(2^{-m}) \leq 2^{-n+1}.$$

In particular, there are a modulus function $g : \mathbb{N} \to \mathbb{N}$ and constants c_0, c_1 such that

$$0 < c_0 \leq \liminf_{n\to\infty} \frac{h(2^{-g(n)})}{2^{-n}} \leq \limsup_{n\to\infty} \frac{h(2^{-g(n)})}{2^{-n}} \leq c_1$$

If, moreover, $h : \mathbb{Q} \to \mathbb{R}$ is a computable gauge function then also $g : \mathbb{N} \to \mathbb{N}$ can be chosen to be computable.

Exact complexity bound

Theorem (*St*'09)

Let $\varphi : \{0,1\}^* \to \{0,1\}^*$ *be a computable dilution function with modulus function* $g : \mathbb{N} \to \mathbb{N}$. *Then*

$\left|\mathrm{KA}\left(\overline{\varphi}(\xi)[0..g(n)]\right) - \mathrm{KA}\left(\xi[0..n]\right)\right| \leq O(1)$ *for all* $\xi \in \{0,1\}^\omega$ *and all* $n \in \mathbb{N}$.

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Theorem

Let $h : \mathbb{Q} \to \mathbb{R}$ be a computable gauge function such that for all $n \ge n_0$ there is an $m \in \mathbb{N}$ with $2^{-n} < h(2^{-m}) \le 2^{-n+1}$. Then

1. $\mathscr{H}^h(\{\xi : \exists c(\mathrm{KA}(\xi[0..n]) \le_{\mathrm{a.e.}} -\log h(2^{-n}) + c)\}) > 0$, *and*
2. *h is an exact dimension function for the sets*
   $\{\xi : \exists c(\mathrm{KA}(\xi[0..n]) \le_{\mathrm{i.o.}} -\log h(2^{-n}) + c)\}$ *and*
   $\{\zeta : \exists c(\mathrm{KA}(\zeta[0..n]) \le_{\mathrm{a.e.}} -\log h(2^{-n}) + c)\}$.

Functions of the Logarithmic Scale

Definition (Functions of the logarithmic scale)

$$h_{(p_0,\dots,p_k)}(t) = t^{p_0} \cdot \prod_{i=1}^{k} \left(\log^i \frac{1}{t}\right)^{-p_i}$$

where $\log^i t := \max\{1, \underbrace{\log_2 \dots \log_2}_{i \text{ times}} t\}$.

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Definition (Generalised Hausdorff Dimension)

$$\begin{aligned} \dim_{\mathrm{H}}^{(k)} F &:= \sup_{<_{\mathrm{lex}}} \{(p_0,\ldots,p_k) : \mathcal{H}^{h_{(p_0,\ldots,p_k)}}(F) = \infty\} \\ &= \inf_{<_{\mathrm{lex}}} \{(p_0,\ldots,p_k) : \mathcal{H}^{h_{(p_0,\ldots,p_k)}}(F) = 0\} \end{aligned}$$

Upper bound

Let $h_{(p_0,\dots,p_k)}$, $k > 0$, be a function of the logarithmic scale. We define $\beta_h := \log h_{(p_0,\dots,p_{k-1})}$. Observe that

$$\begin{aligned} \beta_h(2^{-n}) &= p_0 \cdot n - \sum_{i=1}^{k-1} p_i \cdot \log^i n \quad \text{and} \\ \log h_{(p_0,\dots,p_k)}(2^{-n}) &= p_0 \cdot n - \sum_{i=1}^{k-1} p_i \cdot \log^i n - p_k \cdot \log^k n \end{aligned}$$

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Theorem (MIELKE'10)

Let $k \geq 0$, $(p_0,\dots,p_k)$ be a $(k+1)$-tuple and $h_{(p_0,\dots,p_k)}$ be a function of the logarithmic scale. Then

$$\dim_{\mathrm{H}}^{(k)}\left\{\xi : \xi \in \{0,1\}^\omega \wedge \liminf_{n\to\infty} \frac{\mathrm{KA}(\xi[0..n]) - \beta_h(2^{-n})}{\log^k n} < p_k\right\} \leq (p_0,\dots,p_k).$$

Lower bound

Theorem (Mielke'10, St)

Let $k > 0$, $(p_0, \ldots, p_k)$ be a $(k+1)$-tuple where $p_0, \ldots, p_{k-1}$ are computable reals. Then

$$\dim_{\mathrm{H}}^{(k)}\left\{\xi : \xi \in \{0,1\}^{\omega} \wedge \liminf_{n\to\infty} \frac{\mathrm{KA}(\xi[0..n]) - \beta_h(2^{-n})}{\log^k n} < p_k\right\} = (p_0, \ldots, p_k)$$

for $h = h_{(p_0,\ldots,p_k)}$.

Why our bounds don't match

"Inexact" case: approximation of real by computable reals

For every real number α and every $\varepsilon > 0$ there are computable reals α_0, α_1 such that $|\alpha_1 - \alpha_0| < \varepsilon$ and $\alpha_0 \le \alpha \le \alpha_1$.

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Example: logarithmic scale

If there is a computable function $h : \mathbb{Q} \to \mathbb{R}$ such that

$$t^{p_0} \le h(t) \le t^{p_0} \cdot \log \frac{1}{t} \text{ for } t \in (0,1) \cap \mathbb{Q}$$

then p_0 is computable.

Counter-example

RYABKO's theorem is independent of the complexity. That is, we can replace KA by other complexities, e.g. by plain KOLMOGOROV complexity KS.

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Example [Oscillation of the plain complexity KS]

It is known that $\mathrm{KS}(\xi[0..n]) \leq_{\mathrm{i.o.}} n - \log n + O(1)$. Thus

$$\left\{\xi : \xi \in \{0,1\}^\omega \wedge \liminf_{n\to\infty} \frac{\mathrm{KS}(\xi[0..n]) - n}{\log n} < \varepsilon - 1\right\} = \{0,1\}^\omega \text{ for all } \varepsilon > 0$$

but

$$\dim_{\mathrm{H}}^{(1)} \left\{\xi : \xi \in \{0,1\}^\omega \wedge \liminf_{n\to\infty} \frac{\mathrm{KS}(\xi[0..n]) - n}{\log n} < \varepsilon - 1\right\} = 1 >_{\mathrm{lex}} (1, \varepsilon - 1).$$